FINITE ENERGY FOLIATIONS IN THE RESTRICTED THREE-BODY PROBLEM

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Abstract. This paper is about using pseudo-holomorphic curves to study the circular planar restricted three-body problem. The main result states that for mass ratios sufficiently close to $\frac{1}{2}$ and energies slightly above the first Lagrange value, the flow on the regularized component $\mathbb{R}P^3\#\mathbb{R}P^3$ of the energy surface admits a finite energy foliation with three binding orbits, namely two retrograde orbits around the primaries and the Lyapunov orbit in the neck region about the first Lagrange point. This foliation explains the numerically observed homoclinic orbits to the Lyapunov orbits. The critical energy surface is proved to satisfy the strict convexity condition in regularizing elliptic coordinates. This allows for the application of a general abstract result for Reeb vector fields on holed lens spaces, concerning the existence of finite energy foliations with prescribed binding orbits. As a by-product of the convex analysis, Birkhoff's retrograde orbit conjecture is proved for mass ratios sufficiently close to $\frac{1}{2}$ and all energies below the first Lagrange value. This conjecture states that the retrograde orbit bounds a disk-like global surface of section on each regularized component $\mathbb{R}P^3$ of the energy surface.

Contents

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1. Introduction

This paper studies the circular planar restricted three-body problem for energies up to slightly above the first Lagrange value. The main goal is to find finite energy foliations as introduced by Hofer, Wysocki, and Zehnder in [37] so that the binding is formed by Birkhoff's retrograde orbits around the primaries and the Lyapunov orbit near the first Lagrange point. These foliations, called $3-2-3$ foliations, imply the existence of infinitely many periodic orbits and infinitely many homoclinic orbits to the Lyapunov orbit in the regularized $\mathbb{R}P^3\#\mathbb{R}P^3$-component of the energy surface. The existence of homoclinic orbits to the Lyapunov orbits near the first Lagrange point leads to important applications in the space mission design, see [51].

Before introducing the restricted three-body problem and stating the main results, we briefly explain the abstract setting. Consider a Reeb vector field on a contact three-manifold $\mathcal{M}$ with non-empty boundary $\partial\mathcal{M}$. Assume that each boundary component $\mathcal{S}\subset\partial\mathcal{M}$ consists of a regular two-sphere containing a hyperbolic periodic orbit $P_2\subset\mathcal{S}$, the so-called Lyapunov orbit. The subscript 2 in the notation refers to the Conley-Zehnder index of the Lyapunov orbit. The vector field is transverse and points towards opposite directions along the hemispheres of $\mathcal{S}\setminus P_2$. The interior of $\mathcal{M}$ contains a special periodic orbit P_3, referred to as the retrograde orbit. The notation means that the generalized Conley-Zehnder index of a certain contractible cover of P_3 is at least 3. This is a common scenario in classical Hamiltonian systems near critical energy surfaces that contain saddle-center equilibrium points. For energies slightly above the critical value, a subset $\mathcal{M}$ of the energy surface has boundary components in the neck region about the equilibrium points as above and is referred to as a chamber. One may ask whether a chamber $\mathcal{M}$ contains periodic orbits or homoclinic and heteroclinic orbits connecting the Lyapunov orbits. In the circular planar restricted three-body problem, for energies slightly above the first Lagrange value, the regularized component near the primaries is diffeomorphic to the connected sum $\mathbb{R}P^3\#\mathbb{R}P^3$ and the chamber $\mathcal{M}$ around each primary is diffeomorphic to the real projective three-space $\mathbb{R}P^3\equiv L(2,1)$ with an open ball removed. Each chamber contains a retrograde orbit around the corresponding primary. Thus, the boundary $\partial\mathcal{M}$ of each chamber consists of a single component containing the Lyapunov orbit near the first Lagrange point. In our most general setting, $\mathcal{M}$ is a lens space $L(p,q)$ with finitely many balls removed, and equipped with the standard contact structure.

This paper presents necessary and sufficient conditions for the Lyapunov orbits in $\partial\mathcal{M}$ and the retrograde orbit in $\mathcal{M}\setminus\partial\mathcal{M}$ to form the binding of a finite energy foliation whose leaves are as simple as possible, i.e., planes asymptotic to the retrograde orbit and cylinders connecting the retrograde orbit to each Lyapunov orbit.

We shall apply the main abstract results to the circular planar restricted three-body problem for mass ratios sufficiently close to $\frac{1}{2}$ and energies slightly above the first Lagrange value. The regularized chamber around each primary fits the abstract assumptions due to some convexity estimates on the critical energy surface. More precisely, all periodic orbits that might obstruct the existence of the desired foliation do not exist. In this situation, the chamber around each primary admits a finite energy foliation, which determines a $3-2-3$ foliation for $\mathbb{R}P^3\#\mathbb{R}P^3$.

Our methods consist of studying pseudo-holomorphic curves in the symplectization of the energy surface. The pseudo-holomorphic curves germinate from a Bishop family of pseudo-holomorphic disks with special boundary conditions as in [29]. The theory of pseudo-holomorphic curves in symplectizations was fundamentally developed by Hofer, Wysocki, and Zehnder in [29, 30, 31, 32, 33, 34, 35, 36, 37], and was first applied to the restricted three-body problem in the seminal paper [1].

1.1. **The setting.** Let $\mathcal{M} = \mathcal{M}^3$ be a smooth three-manifold diffeomorphic to a lens space $L(p, q)$ with finitely many disjoint regular open balls removed. Recall that $p \geq q \geq 1$ are relatively prime and $L(p, q)$ is the quotient of $S^3 = \{(z_1, z_2) \in \mathbb{C} \times \mathbb{C} : |z_1|^2 + |z_2|^2 = 1\}$ by the free $\mathbb{Z}_p$-action generated by $(z_1, z_2) \mapsto (e^{2\pi i/p} z_1, e^{2\pi iq/p} z_2)$.

The boundary components $\mathcal{S}_i \subset \partial\mathcal{M}, i = 1, \ldots, l$, are diffeomorphic to the two-sphere. Let α be a contact form on $\mathcal{M}$ defining the standard contact structure of $L(p, q)$. Assume that the Reeb vector field $\mathcal{R}$ of α admits a hyperbolic periodic orbit $P_{2,i} \subset \mathcal{S}_i$ for each i, and $\mathcal{R}$ is transverse and points towards opposite directions along the hemispheres of $\mathcal{S}_i \setminus P_{2,i}$. The notation indicates that the Conley-Zehnder index of $P_{2,i}$ is 2 in a natural frame of the contact structure $\xi = \ker \alpha$ along $\mathcal{S}_i$. Let $P_3 \subset \mathcal{M} \setminus \partial\mathcal{M}$ be a p-unknotted periodic orbit with self-liking number $-1/p$, i.e., there exists an immersed closed disk $\mathcal{D} \hookrightarrow \mathcal{M} \setminus \partial\mathcal{M}$ whose interior is embedded, and whose boundary p-covers P_3. The disk $\mathcal{D}$ is called a p-disk for P_3. The characteristic foliation $(T\mathcal{D} \cap \xi)^\perp$ has a unique singularity in the interior of $\mathcal{D}$, which is positive and nicely elliptic. Denote by P_3^j the j-cover of P_3 for every positive integer j, and let $\mathcal{P}$ be the set of periodic orbits $P_3, P_{2,1}, \ldots, P_{2,l}$, called binding orbits. The following definition is based on [17, 63].

Definition 1.1. *A weakly convex foliation $\mathcal{F}$ of $\mathcal{M}$ adapted to α and $\mathcal{P}$ is a singular foliation of $\mathcal{M}$ whose singular set is formed by $\cup_{P \in \mathcal{P}} P$, and $\mathcal{M} \setminus \cup_{P \in \mathcal{P}} P$ is foliated by regular surfaces transverse to the Reeb vector field $\mathcal{R}$. Each regular leaf $\dot{\Sigma} \hookrightarrow \mathcal{M} \setminus \cup_{P \in \mathcal{P}} P$ of $\mathcal{F}$ is a properly embedded punctured two-sphere $\dot{\Sigma} \equiv \mathbb{C}P^1 \setminus \Gamma$, $0 < \#\Gamma < +\infty$. At each puncture $z \in \Gamma$, $\dot{\Sigma}$ is asymptotic to some $P_{2,i} \in \mathcal{P}$ or to P_3^p. This orbit, denoted P_z, is called the asymptotic limit of $\dot{\Sigma}$ at z. The closure $\Sigma \subset \mathcal{M}$ of $\dot{\Sigma}$ is an immersion. We call the puncture $z \in \Gamma$ positive if the boundary orientation of P_z induced by the orientation of $\dot{\Sigma}$ coincides with the flow orientation. The puncture z is called negative, otherwise. Here, $\mathcal{M}$ is oriented by $\alpha \wedge d\alpha > 0$ and $\dot{\Sigma}$ is co-oriented by $\mathcal{R}$. The asymptotic limits of $\dot{\Sigma}$ are distinct, and precisely one puncture of $\dot{\Sigma}$ is positive. If $\dot{\Sigma}$ is asymptotic to P_3^p at $z \in \Gamma$, then z is positive and all other punctures in $\Gamma \setminus \{z\}$ are negative and asymptotic to distinct $P_{2,j}$. If P_3^p is not an asymptotic limit of $\dot{\Sigma}$, then $\dot{\Sigma} \equiv \mathbb{C}P^1 \setminus \{\infty\}$ and $\dot{\Sigma} \subset \partial\mathcal{M}$ is a hemisphere of $\mathcal{S}_j \setminus P_{2,j}$ for some j. The hemispheres of $\mathcal{S}_i \setminus P_{2,i}, i = 1, \ldots l$, are regular leaves of $\mathcal{F}$.*

The weakly convex foliations $\mathcal{F}$ we shall construct are projections to $\mathcal{M}$ of finite energy foliations $\tilde{\mathcal{F}}$ in the symplectization $\mathbb{R} \times \mathcal{M}$. The leaves of $\tilde{\mathcal{F}}$ are the image of embedded finite energy J-holomorphic curves associated with almost complex structures J on $\mathbb{R} \times \mathcal{M}$ adapted to α. For generic choices of J, we shall construct $\tilde{\mathcal{F}}$ whose projection $\mathcal{F}$ to $\mathcal{M}$ is as simple as possible, i.e., the regular leaves that are asymptotic to P_3^p at a positive puncture have at most one negative puncture asymptotic to some $P_{2,j}$. This means that the regular leaves are planes asymptotic to P_3^p, cylinders connecting P_3^p at the positive puncture to $P_{2,j}$ at the negative puncture, and planes projecting to the hemispheres of $\mathcal{S}_j \setminus P_{2,j}$ for each j. The planes asymptotic to P_3^p lie in a one-parameter family of planes with the same asymptotic limit, and there exist precisely l such families. Also, there exists a unique rigid cylinder from P_3^p to each $P_{2,j}, j = 1, \ldots, l$. The rigid cylinders separate the l families of planes asymptotic to P_3^p, i.e., each family planes asymptotic to P_3^p breaks at each of its ends onto a rigid cylinder connecting P_3^p to $P_{2,j}$ and a hemisphere of $\mathcal{S}_j \setminus P_{2,j}$.

The $\mathbb{R}$-invariant almost complex structure J on the symplectization $\mathbb{R} \times \mathcal{M}$ preserves the contact structure $\xi = \ker \alpha$, where it is $d\alpha$-compatible, and sends ∂_a to the Reeb vector field $\mathcal{R}$. In this way, J is compatible with the symplectic form $d(e^a \alpha)$ on $\mathbb{R} \times \mathcal{M}$, where a is the $\mathbb{R}$-coordinate. The leaves of $\tilde{\mathcal{F}}$ are the image of J-holomorphic curves $\tilde{u} = (a, u) : \mathbb{C}P^1 \setminus \Gamma \to \mathbb{R} \times \mathcal{M}$, $0 < \#\Gamma < +\infty$, with finite Hofer's energy $0 < E(\tilde{u}) < +\infty$. A J-holomorphic cylinder whose image is $\mathbb{R} \times P$ for some periodic orbit P, is called a trivial cylinder over P. An embedded J-holomorphic curve whose projection to $\mathcal{M}$ is also embedded is called nicely embedded [69, 70, 72].

Definition 1.2. *A finite energy foliation $\tilde{\mathcal{F}}$ adapted to α, $\mathcal{P}$ and J is a regular foliation $\tilde{\mathcal{F}}$ of $\mathbb{R}\times\mathcal{M}$ whose leaves are trivial cylinders over the binding orbits or the image of nicely embedded J-holomorphic curves with uniformly bounded energies. The projection of $\tilde{\mathcal{F}}$ to $\mathcal{M}$ is a weakly convex foliation $\mathcal{F}$ adapted to α and $\mathcal{P}$ as in Definition 1.1. The trivial cylinders over the binding orbits are leaves of $\tilde{\mathcal{F}}$, and $F+a\in\tilde{\mathcal{F}}$ for every $F\in\tilde{\mathcal{F}}$ and $a\in\mathbb{R}$.*

1.2. **The main abstract results.** Let us assume that the contact form α on $\mathcal{M}$ is the restriction to $\mathcal{M}$ of a contact form on $L(p,q)$ equipped with the standard tight contact structure $\xi=\xi_{p,q}$. Recall that $\xi_{p,q}$ is induced by the $\mathbb{Z}_p$-invariant contact form λ_0 on S^3 given by the restriction of the Liouville form $\frac{1}{4i}\sum_{j=1}^2(\bar{z}_j dz_j - z_j d\bar{z}_j)$ to $S^3\subset\mathbb{C}\times\mathbb{C}$. Denote by $\pi_{p,q}:S^3\to L(p,q)$ the natural projection. The projections to $L(p,q)$ of $S^1\times\{0\}$ and $\{0\}\times S^1$ are p-unknotted, denoted K_1 and K_2, respectively. The disk $u_1(z)=(z,\sqrt{1-|z|^2}), z\in\mathbb{D}$, projects to a p-disk for K_1. Similarly, $u_2(z)=(\sqrt{1-|z|^2},z), z\in\mathbb{D}$, projects to a p-disk for K_2. We view $[K_1]\equiv 1$ as a generator of $\pi_1(L(p,q))\simeq\mathbb{Z}_p$. In that case, $[K_2]=q^*$, where $qq^*\equiv 1 \mod p$. A local transverse section to K_1 determines p distinct branches of u_1 approaching K_1. Along the positive period of K_1, the branches are permuted by $-q$, where the orientation is induced by $d\alpha$. The number $-q$ is called the monodromy of K_1. It does not depend on the p-disk for K_1. Similarly, one checks that the monodromy of K_2 is $-q^*$, see [40, Lemma 3.5]. Notice that $q^*=1$ and $[K_1]=[K_2]$ in the case of $L(2,1)\equiv\mathbb{R}P^3$.

The lift of P_3 to S^3 under the natural projection $\pi_{p,q}$ is a $\mathbb{Z}_p$-symmetric trivial knot $K\subset S^3\equiv L(1,1)$ transverse to the standard tight contact structure $\xi=\xi_{1,1}$ which is transversely isotopic to the Hopf fiber $S^1\times\{0\}\subset S^3$. We say that a periodic orbit $P'\subset\mathcal{M}\setminus(\partial\mathcal{M}\cup P_3)$ is linked with P_3 if the algebraic intersection number $P'\cdot\mathcal{D}\in\mathbb{Z}$ does not vanish, i.e., $\mathrm{link}(P',P_3^p)\neq 0$, where $\mathcal{D}$ is a p-disk for P_3. Otherwise, we say that P' is unlinked with P_3. Let

$$\mathcal{S}(\mathcal{D},\alpha):=\int_{\mathcal{D}}|d\alpha|.$$

We call $\mathcal{S}(\mathcal{D},\alpha)$ the $|d\alpha|$-area of $\mathcal{D}$. Notice that if $\mathcal{D}$ is transverse to the Reeb vector field, then $\mathcal{S}(\mathcal{D},\alpha)$ coincides with the action of P_3^p. The set of periodic orbits of α is denoted by $\mathcal{P}(\alpha)$, and an element of $\mathcal{P}(\alpha)$ is a pair $P=(x,T)$, where $x:\mathbb{R}/T\mathbb{Z}$ satisfies $\dot{x}=\mathcal{R}(x)$, and $T>0$ is a period of x. Two such pairs are identified if they have the same image and the same period. Iterates of P are denoted by $P^p=(x,pT)\in\mathcal{P}(\alpha)$ for every $p\in\mathbb{N}$. The action of P coincides with its period and is denoted by $\mathcal{A}(P)=\int_P\alpha=\int_{\mathbb{R}/T\mathbb{Z}}x^*\alpha=T$. The rotation number of a contractible periodic orbit $P\in\mathcal{P}(\alpha)$ is denoted by $\rho(P)$. This is a well-defined real number computed in a global symplectic frame of $\xi\to S^3$. The Lyapunov orbits are contractible and satisfy $\rho(P_{2,j})=1$ for every j. We always assume that the contractible p-th iterate of the retrograde orbit P_3 satisfies $\rho(P_3^p)>1$.

We assume the existence of an almost complex structure J on $\mathbb{R}\times\mathcal{M}$ so that the hemispheres of $\mathcal{S}_j\setminus P_{2,j}, j=1,\dots,l$, are projections of nicely embedded J-holomorphic planes asymptotic to $P_{2,j}$. Such planes are unique and they always exist in our concrete applications.

The main abstract result establishes sufficient conditions for the existence of a finite energy foliation projecting to a weakly convex foliation adapted to α and $\mathcal{P}$.

Theorem 1.3. *Assume that $\mathcal{M}$, α, $\mathcal{P}$ and J satisfy the above conditions. Let $\mathcal{P}'\subset\mathcal{P}(\alpha)$ be the set of contractible periodic orbits $P'\subset\mathcal{M}\setminus(\partial\mathcal{M}\cup P_3)$ satisfying*

$$\rho(P')=1,\quad \mathrm{link}(P',P_3^p)=0\quad \textit{and}\quad \mathcal{A}(P')\leq\mathcal{S}(\mathcal{D},\alpha). \tag{1.1}$$

If $\mathcal{P}'=\emptyset$, then there exists a finite energy foliation $\tilde{\mathcal{F}}$ adapted to α, $\mathcal{P}$, and J' whose leaves are planes and cylinders. In particular, the projection $\mathcal{F}$ of $\tilde{\mathcal{F}}$ to $\mathcal{M}$ is a weakly convex foliation. Here, the almost complex structure J' is C^∞-arbitrarily close to J and coincides with J on a neighborhood of $\partial\mathcal{M}$.

The construction of a finite energy foliation as in Theorem 1.3 starts from a Bishop family of J-holomorphic disks with boundary on $\{0\}\times\mathcal{D}$, where $\mathcal{D}$ is a p-disk for P_3 whose characteristic foliation has a unique singularity, which is nicely elliptic. This analysis was first introduced by

Hofer in [29] in his proof of the Weinstein conjecture for overtwisted contact three-manifolds, and later developed by Hofer, Wysocki, and Zehnder in [31, 32]. The boundary of the disks in the Bishop family is radially monotone from the nicely elliptic singularity towards the boundary $\partial\mathcal{D} \equiv P_3^p$. We may first assume that the contact form α is nondegenerate and J satisfies some generic conditions to be specified later. If no bubbling-off occurs before the boundary of the disks in the Bishop family approaches P_3^p, then the family ultimately converges in the SFT sense to a building containing J-holomorphic curves asymptotic to orbits in $\mathcal{P}$, in particular to P_3^p. More precisely, discarding the trivial half-cylinder over P_3^p at the highest level of the building, the SFT-limit of the Bishop family is either a nicely embedded J-holomorphic plane asymptotic to P_3^p or a cylinder asymptotic to P_3^p at the positive puncture and to some $P_{2,j}$ at the negative puncture, plus a J-holomorphic plane projecting to a hemisphere of $\mathcal{S}_j \setminus P_{2,j}$. By construction, the leading eigenvector of the curve approaching P_3^p is the correct one, and such curves are part of the desired foliation. Moreover, these curves are enough to construct the remaining foliation, obtained as a direct application of the weighted Fredholm theory, the gluing theorem for regular curves, and the SFT compactness theorem. Now, if the Bishop family admits bubbling-off before reaching $\partial\mathcal{D} = P_3^p$, then the compactness theorem implies that the SFT-limit of the Bishop family consists of a half-cylinder with the same boundary conditions on $\{0\} \times \mathcal{D}$ and asymptotic to some $P_{2,j}$ at the negative puncture, plus a J-holomorphic plane projecting to a hemisphere of $\mathcal{S}_j \setminus P_{2,j}$ for some j. In this case, gluing the half-cylinder with the other hemisphere of $S_{2,j} \setminus P_{2,j}$, one obtains a new Bishop family of disks whose boundary is even closer to P_3^p. This monotone continuation eventually leads to a sequence of J-holomorphic disks whose boundary converges to P_3^p, reducing the construction of the desired foliation to the previous case. For this strategy to work, two ingredients are crucial. First, J may need to be perturbed to avoid non-regular curves in the compactness argument, allowing the application of Fredholm theory and the gluing theorem. Second, the uniqueness of disks in the Bishop family, as discussed in [3], ensures that the Bishop family obtained by gluing the half-cylinder with the opposite rigid plane is monotone, i.e., the boundary of the disks continues outside of those in the previous family and thus gets closer to P_3^p.

One of the immediate dynamical consequences of a finite energy foliation, as in Theorem 1.3, is the existence of a homoclinic or a heteroclinic orbit to the Lyapunov orbit with the largest action.

Corollary 1.4. *Under the conditions of Theorem 1.3, let $P_{2,j}$ be the Lyapunov orbit with the largest action. Then $P_{2,j}$ admits a homoclinic orbit, or a heteroclinic orbit to some $P_{2,k}, k \neq j$. In particular, if $l = 1$, then the Lyapunov orbit $P_{2,1} \subset \partial\mathcal{M}$ admits a homoclinic orbit in $\mathcal{M}$.*

1.3. **The main application.** We explain how Theorem 1.3 applies to the circular planar restricted three-body problem for a range of mass ratios and energies slightly above the first Lagrange value and how the $3-2-3$ foliation implies the existence of infinitely many homoclinic orbits to the Lyapunov orbit.

Consider two massive bodies, called primaries, moving along circular trajectories about their fixed center of mass. The massless satellite moves on the same plane as the primaries and is attracted by them according to Newton's gravitational law. The Hamiltonian describing the motion of the satellite in a rotating system that fixes the primaries is given in canonical coordinates by

$$H_\mu(p,q) = \frac{1}{2}|p + iq|^2 - \frac{\mu}{|q-(1-\mu)|} - \frac{1-\mu}{|q+\mu|} - \frac{1}{2}|q|^2.$$

Here, $q = q_1 + iq_2 \in \mathbb{C} \setminus \{-\mu, 1-\mu\}$ is the position of the satellite, and $p = p_1 + ip_2 \in \mathbb{C}$ is the momentum. The mass ratio $0 < \mu < 1$ is the mass of the primary at $1-\mu \in \mathbb{C}$, and $1-\mu$ is the mass of the primary at $-\mu \in \mathbb{C}$. The primaries are called the moon and the earth, respectively.

For each $0 < \mu < 1$, $H = H_\mu$ has precisely five critical points $l_1(\mu), \ldots, l_5(\mu)$, called Lagrange points, with increasing critical values $L_1(\mu) < L_2(\mu) \leq L_3(\mu) < L_4(\mu) = L_5(\mu)$ called Lagrange values. Notice that $L_2(1/2) = L_3(1/2)$ and l_2, l_3 are interchanged as μ crosses $1/2$. The Lagrange points are the rest points of the Hamiltonian flow of H_μ, and $l_1(\mu)$ projects between the earth and the moon. For $E < L_1(\mu)$, the energy surface $H^{-1}(E)$ has three regular components $\mathcal{M}^e_{\mu,E}, \mathcal{M}^m_{\mu,E}, \mathcal{M}^u_{\mu,E}$, the first two components project to punctured disk-like domains about the earth and the moon, respectively, and the third one projects to a neighborhood of ∞. If the energy E coincides with the first Lagrange value $L_1(\mu)$, then $\mathcal{M}^e_{\mu,L_1(\mu)}$ and $\mathcal{M}^m_{\mu,L_1(\mu)}$ touch each other at $l_1(\mu)$, a common singularity. Their projections to the q-plane touch each other at a common boundary singularity. For energies $L_1(\mu) < E < L_2(\mu)$, the energy surface has a regular component $\mathcal{M}^{e\#m}_{\mu,E}$ corresponding to the connected sum of $\mathcal{M}^e_{\mu,E}$ and $\mathcal{M}^m_{\mu,E}$. It projects to a twice-punctured disk-like domain about the earth and the moon.

The components projecting near the primaries are unbounded in the p-direction and contain trajectories that collide with the primaries in finite time. We regularize collisions using elliptic coordinates. The regularized Hamiltonian in symplectic coordinates $(y, x) \in \mathbb{R}^2 \times (\mathbb{R} \times \mathbb{R}/2\pi\mathbb{Z})$ has the form $\hat{H}_{\mu,E} = \frac{1}{2}|y + F(x)|^2 + V_{\mu,E}(x)$, where $F(x) = (f_1(x_2), f_2(x_1))$ and $V_{\mu,E}(x)$ are smooth functions on $\mathbb{R} \times \mathbb{R}/2\pi\mathbb{Z}$, with $V_{\mu,E}$ smoothly depending on (μ, E). Up to a time reparametrization, the dynamics on $H^{-1}_\mu(E)$ corresponds to the dynamics on $\hat{H}^{-1}_{\mu,E}(0)$ under a double covering map with antipodal symmetry. The components of the energy surface quotiented by the antipodal symmetry now become smooth compact submanifolds

$$
\begin{aligned}
&S^3 \overset{2:1}{\to} \mathcal{M}^e_{\mu,E}, \mathcal{M}^m_{\mu,E} \equiv \mathbb{R}P^3, && \forall E < L_1(\mu), \\
&S^1 \times S^2 \overset{2:1}{\to} \mathcal{M}^{e\#m}_{\mu,E} \equiv \mathbb{R}P^3 \# \mathbb{R}P^3, && \forall L_1(\mu) < E < L_2(\mu).
\end{aligned}
\tag{1.2}
$$

For simplicity, we keep the same notation for the regularized components.

In [1], Albers, Frauenfelder, van Koert, and Paternain observed that for energies up to slightly above $L_1(\mu)$, these regularized energy surfaces have contact type. In particular, the methods of pseudo-holomorphic curves apply. For energies below $L_1(\mu)$, the flow is equivalent to a Reeb flow on $\mathbb{R}P^3$ equipped with the universally tight contact structure ξ_0. Birkhoff [6] used the shooting method to prove the existence of a retrograde orbit, i.e., a periodic orbit projecting to a simple closed curve around the primary moving opposite to the rotating system. He raised the question of whether the retrograde orbit bounds a disk-like global surface of section.

As mentioned above, for every E slightly above $L_1(\mu)$, the regularized flow on $\mathcal{M}^{e\#m}_{\mu,E}$ is equivalent to the Reeb flow of a contact form $\alpha = \alpha_{\mu,E}$ on the contact connected sum $(\mathbb{R}P^3 \# \mathbb{R}P^3, \xi_0 \# \xi_0)$. In the neck region of the connected sum, there exists a low action index-2 hyperbolic orbit $P_2 = P_{2,\mu,E}$, called the Lyapunov orbit. This orbit bounds a pair of closed disks whose interior is transverse to the flow. They form a regular two-sphere $\mathcal{S}$ which separates $\mathcal{M}^{e\#m}_{\mu,E}$ into two components whose closures, denoted by $\mathcal{M}^e_{\mu,E}$ and $\mathcal{M}^m_{\mu,E}$, are contactomorphic to $(\mathbb{R}P^3, \xi_0)$ with an open ball removed. One can prove using the same argument as Birkhoff that the interiors of $\mathcal{M}^e_{\mu,E}$ and $\mathcal{M}^m_{\mu,E}$ possess retrograde orbits P^e_3 and P^m_e, respectively, see Theorem 1.7. They are 2-unknotted, and their self-linking number is $-1/2$. This means that they admit 2-disks whose characteristic foliation has a unique nicely elliptic singularity.

We aim at finding a weakly convex foliation on $\mathcal{M}^{e\#m}_{\mu,E}$ whose binding is formed by the Lyapunov orbit P_2 and the retrograde orbits P^e_3, P^m_3. Firstly, we observe that there is no canonical choice of a separating two-sphere $\mathcal{S}$ containing the Lyapunov orbit P_2. We fix an almost complex structure J on the symplectization of $\mathcal{M}^{e\#m}_{\mu,E}$ and require that the hemispheres of $\mathcal{S}$ are projections of J-holomorphic planes asymptotic to P_2. We also require them to be in the neck region and whose distance to $l_1(\mu)$ tend to 0 as $E \to L_1(\mu)^+$.

The following proposition generalizes the results in [1]. It provides a choice of α and J, and thus gives a precise definition of the separating two-sphere $\mathcal{S}$ as the projection of J-holomorphic curves, so that $\mathcal{S}$ is arbitrarily close to $l_1(\mu)$.

Theorem 1.5. *Let $0 < \mu_0 < 1$. Then for every (μ, E) sufficiently close to $(\mu_0, L_1(\mu_0))$, with $E > L_1(\mu)$, the following statements hold:*

(i) *There exists a contact form $\alpha = \alpha_{\mu,E} = i_Y \omega_0$ on $\mathcal{M}^{e\#m}_{\mu,E} \equiv \mathbb{R}P^3 \# \mathbb{R}P^3$ whose Reeb flow is equivalent to the regularized Hamiltonian flow. Here, $Y = Y_{\mu,E}$ is a Liouville vector field, defined on a neighborhood of $\mathcal{M}^{e\#m}_{\mu,E}$ in $\mathbb{R}^4$ and transverse to $\mathcal{M}^{e\#m}_{\mu,E}$. Also, ω_0 is the canonical symplectic form $\sum_i dp_i \wedge dq_i$.*

(ii) *There exists a compatible almost complex structure $J = J_{\mu,E}$ on $\mathbb{R} \times \mathcal{M}^{e\#m}_{\mu,E}$ adapted to α that admits a pair of J-holomorphic planes asymptotic to $P_{2,E}$ through opposite directions.*

The closure of their projections to $\mathcal{M}^{e\#m}_{\mu,E}$ form a regular two-sphere $\mathcal{S} = \mathcal{S}_{\mu,E}$ containing $P_{2,E}$. Furthermore, dist$(\mathcal{S}, l_1(\mu)) \to 0$ as $E \to L_1(\mu)^+$ uniformly in μ.

(iii) *There exists a contact form $\alpha = \alpha_{\mu,L_1(\mu)} = i_{Y_{\mu,L_1(\mu)}}\omega_0$ on the sphere-like singular subset $\dot{\mathcal{M}}^e := \mathcal{M}^e_{\mu,L_1(\mu)} \setminus \{l_1(\mu)\}$ so that the contact forms $\alpha_{\mu,E}$ in (i) converge in $C^\infty_{loc}(\dot{\mathcal{M}}^e)$ to $\alpha_{\mu,L_1(\mu)}$ as $E \to L_1(\mu)^+$ uniformly in μ. The same conclusion holds for $\dot{\mathcal{M}}^m := \mathcal{M}^m_{\mu,L_1(\mu)} \setminus \{l_1(\mu)\}$.*

We consider the contact form α on $\mathcal{M}^{e\#m}_{\mu,E}$ and the almost complex structure J on $\mathbb{R} \times \mathcal{M}^{e\#m}_{\mu,E}$ as in Theorem 1.5. The regular two-sphere $\mathcal{S}$ separates $\mathcal{M}^{e\#m}_{\mu,E}$ into two compact subsets $\mathcal{M}^e_{\mu,E}$ and $\mathcal{M}^m_{\mu,E}$, called chambers. As mentioned before, each chamber has at least one retrograde orbit. We fix them and denote them by P_3^e and P_3^m, respectively. We have $P_2 \subset \mathcal{S} = \partial\mathcal{M}^e_{\mu,E} = \partial\mathcal{M}^m_{\mu,E}$, $P_3^e \subset \mathcal{M}^e_{\mu,E} \setminus \mathcal{S}$ and $P_3^m \subset \mathcal{M}^m_{\mu,E} \setminus \mathcal{S}$.

We aim to prove the existence of finite energy foliations projecting to special weakly convex foliations of $\mathcal{M}^e_{\mu,E}\#\mathcal{M}^m_{\mu,E}$, called $3-2-3$ foliations.

Definition 1.6. *Consider the following terminology from* [15]*:*

(i) *A $2-3$ foliation of $\mathcal{M}^e_{\mu,E}$ adapted to α, J and $\mathcal{P}^e = \{P_3^e, P_2\}$, is a weakly convex foliation $\mathcal{F}^e$ of $\mathcal{M}^e_{\mu,E}$ adapted to α and $\mathcal{P}^e$. The regular leaves consist of the hemispheres U_1, U_2 in $\mathcal{S} \setminus P_{2,E}$, a one-parameter family of planes asymptotic to $(P_3^e)^2$ and a rigid cylinder with a positive end at $(P_3^e)^2$ and a negative end at P_2. They are transverse to the flow and consist of projections to $\mathcal{M}^e_{\mu,E}$ of a finite energy foliation in the symplectization.*

(ii) *A $2-3$ foliation $\mathcal{F}^m$ of $\mathcal{M}^m_{\mu,E}$ adapted to α, J and $\mathcal{P}^m = \{P_3^m, P_2\}$ is defined similarly to (i).*

(iii) *If $2-3$ foliations $\mathcal{F}^e$ and $\mathcal{F}^m$ of $\mathcal{M}^e_{\mu,E}$ and $\mathcal{M}^m_{\mu,E}$ exist, respectively, then $\mathcal{F}^e \cup \mathcal{F}^m$ is called a $3-2-3$ foliation of $\mathcal{M}^{e\#m}_{\mu,E}$ adapted to α, J and $\mathcal{P} = \{P_3^e, P_3^m, P_2\}$.*

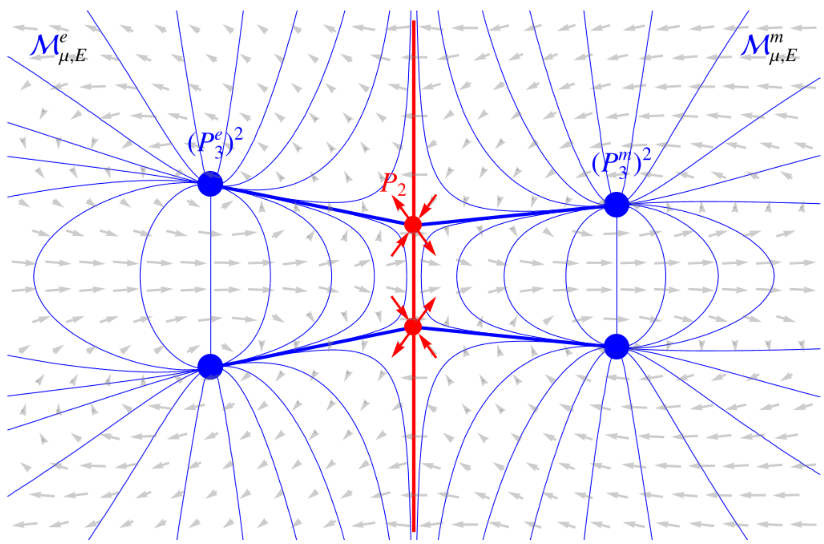

FIGURE 1.1. The $3-2-3$ foliation on the regularized component $\mathcal{M}^{e\#m}_{\mu,E} \equiv \mathbb{R}P^3\#\mathbb{R}P^3$ for mass ratios close to $1/2$ and energies slightly above the first Lagrange value. The rigid cylinders (bold blue) connect the double cover of the retrograde orbits P_3^e and P_3^m to the Lyapunov orbit P_2 near the first Lagrange point. The rigid planes asymptotic to P_2 (bold red) separate $\mathcal{M}^e_{\mu,E}$ and $\mathcal{M}^m_{\mu,E}$.

The orbit P_3^e or P_3^m in the definition above can be replaced with any 2-unknotted periodic orbit with self-linking number $-1/2$. As already mentioned, Birkhoff [6] proved the existence of a retrograde orbit in $\mathcal{M}^e_{\mu,E}$ for energies below $L_1(\mu)$. A simple argument allows one to extend this result for energies up to slightly above $L_1(\mu)$.

Theorem 1.7. *Fix $0<\mu_0<1$ and $E_0 \leq L_1(\mu_0)$. The following statements hold for every (μ,E) sufficiently close to (μ_0,E_0):*

(i) *There exists a continuous family of retrograde orbits $P_3^e = P^e_{3,\mu,E} \subset \mathcal{M}^e_{\mu,E}$, whose projection to the q-plane is a simple closed curve rotating counterclockwise around the Earth.*

(ii) *$P^e_{3,\mu,E} \to P^e_{3,\mu_0,E_0}$ in C^∞ as $(\mu,E)\to(\mu_0,E_0)$. In particular, the action of P_3^e is uniformly bounded, and there exists a continuous family of 2-disks $\mathcal{D}=\mathcal{D}_{\mu,E}\subset \mathcal{M}^e_{\mu,E}$ for P_3^e whose $|d\alpha|$-area $\mathcal{S}(\mathcal{D},\alpha)$ is uniformly bounded.*

A similar result holds for $\mathcal{M}^m_{\mu,E}$.

The existence of a retrograde orbit, as in Theorem 1.7, follows from the usual Birkhoff's shooting method. Indeed, this method considers trajectories issuing perpendicularly from certain open intervals in the q_1-axis containing the primary as an endpoint. Such intervals are shown to be uniformly away from $l_1(\mu)$ as $E\to L_1(\mu)$ and, moreover, they parametrize two non-self-intersecting real-analytic curves in the interior of the rectangle $Q:=[-\pi/2,\pi/2]\times[-M,0]$, for some $M>0$. At the endpoints, these curves tend to ∂Q. An intersection point between these curves correspond to a retrograde orbit. A topological crossing of such curves always exists and persists under small perturbations of (μ,E), thus giving at least one family of retrograde orbits near (μ_0,E_0) as in the statement.

Even though we denote the retrograde orbit by P_3, we have not yet established that the index of its contractible double cover is greater than or equal to 3. We, therefore, introduce the following definitions.

Definition 1.8. *Let $0<\mu<1$, and let $\alpha=\alpha_{\mu,L_1(\mu)}$ be the contact form on the regularized critical subset $\mathcal{M}^e_{\mu,L_1(\mu)}$ as in Theorem 1.5-(iii).*

(i) *We say that $\mathcal{M}^e_{\mu,L_1(\mu)}$ is dynamically convex if the Conley-Zehnder index of every contractible periodic orbit of α is at least 3.*

(ii) *Let $\widetilde{\mathcal{M}}^e_{\mu,L_1(\mu)}\subset\mathbb{R}^4$ be a component of the lift of the regularized critical subset $\mathcal{M}^e_{\mu,L_1(\mu)}$ to $\mathbb{R}^4$, which double covers $\mathcal{M}^e_{\mu,L_1(\mu)}$. In particular, $\widetilde{\mathcal{M}}^e_{\mu,L_1(\mu)}$ is a topological three-sphere with antipodal symmetry and two opposite saddle-center singularities $S_\pm$ corresponding to the first Lagrange point $l_1(\mu)$. We say that $\mathcal{M}^e_{\mu,L_1(\mu)}$ is strictly convex if $\widetilde{\mathcal{M}}^e_{\mu,L_1(\mu)}$ bounds a convex subset of $\mathbb{R}^4$ and all sectional curvatures of $\widetilde{\mathcal{M}}^e_{\mu,L_1(\mu)}\setminus\{S_\pm\}$ are positive. A similar definition holds for $\mathcal{M}^m_{\mu,L_1(\mu)}$.*

Before stating the main criterion for the existence of $3-2-3$ foliations for energies slightly above the first Lagrange value, we prove a crucial estimate on the index of periodic orbits, which states that periodic orbits passing sufficiently close to $l_1(\mu)$ must have a high Conley-Zehnder index.

Theorem 1.9. *Let $0<\mu_0<1$. Given $N\in\mathbb{N}$, there exists an open neighborhood $\mathcal{U}_N\subset\mathbb{R}^2\times(\mathbb{R}\times\mathbb{R}/2\pi\mathbb{Z})$ of the saddle-center singularities $S_\pm(\mu_0)$, corresponding to the first Lagrange point $l_1(\mu_0)$, such that for every (μ,E) sufficiently close to $(\mu_0,L_1(\mu_0))$, the following holds: if $P'\subset\hat{H}^{-1}_{\mu,E}(0)$ is a periodic orbit that is not a cover of the Lyapunov orbit near $l_1(\mu)$ and $P'\cap\mathcal{U}_N\neq\emptyset$, then $\mu_{CZ}(P')>N$.*

The following theorem states that the regularized component $\mathcal{M}^{e\#m}_{\mu,E}$ admits a $3-2-3$ foliation for E slightly above $L_1(\mu)$ provided some sufficient conditions on the regularized critical subsets $\mathcal{M}^e_{\mu,L_1(\mu)},\mathcal{M}^m_{\mu,L_1(\mu)}$ are satisfied.

Theorem 1.10. *Let $0<\mu_0<1$, and let $\alpha=\alpha_{\mu,E}$, $J=J_{\mu,E}$ and $\mathcal{S}=\partial\mathcal{M}^e_{\mu,E}=\partial\mathcal{M}^m_{\mu,E}$ be defined for (μ,E) sufficiently close to $(\mu_0,L_1(\mu_0))$, with $E>L_1(\mu)$, as in Theorem 1.5. Let $P_3^e=P^e_{3,\mu,E}\subset\mathcal{M}^e_{\mu,E}$ be the continuous family of retrograde orbits and $\mathcal{D}=\mathcal{D}_{\mu,E}$ be the 2-disk for*

P_3^e as given in Theorem 1.7 for (μ,E) sufficiently close to $(\mu_0, L_1(\mu_0))$. Let $\mathcal{P}_0' \subset \mathcal{P}(\alpha_{\mu_0,L_1(\mu_0)})$ be the set of contractible periodic orbits $P' \subset \mathcal{M}^e_{\mu_0,L_1(\mu_0)} \setminus (P_{2,\mu_0,L_1(\mu_0)} \cup P^e_{3,\mu_0,L_1(\mu_0)})$ satisfying

$$\rho(P') = 1, \quad \mathrm{link}(P', P^e_{3,\mu_0,L_1(\mu_0)}) = 0 \quad \text{and} \quad \mathcal{A}(P') \leq \mathcal{S}(\mathcal{D}_{\mu_0,L_1(\mu_0)}, \alpha_{\mu_0,L_1(\mu_0)}).$$

Assume that $\mu_{CZ}((P^e_{3,\mu_0,L_1(\mu_0)})^2) \geq 3$ and $\mathcal{P}_0' = \emptyset$. Then the following assertions hold for every (μ,E) sufficiently close to $(\mu_0,L_1(\mu_0))$, with $E > L_1(\mu)$:

(i) *The index-2 Lyapunov orbit $P_2 \subset \mathcal{S}$ is the unique contractible periodic orbit unlinked with P_3^e in $\mathcal{M}^e_{\mu,E}$, with rotation number 1 and action $\leq \mathcal{S}(\mathcal{D},\alpha)$.*
(ii) *$\mathcal{M}^e_{\mu,E}$ admits a $2-3$ foliation whose binding orbits are the retrograde orbit P_3^e and the Lyapunov orbit P_2 around the first Lagrange point $l_1(\mu)$.*
(iii) *$\mathcal{M}^e_{\mu,E}$ admits infinitely many periodic orbits and infinitely many homoclinic orbits to the Lyapunov orbit near $l_1(\mu)$. Moreover, if the branches in $\mathcal{M}^e_{\mu,E}$ of the stable and unstable manifolds of the Lyapunov orbit do not coincide, then the topological entropy of the flow on $\mathcal{M}^e_{\mu,E}$ is positive.*

A similar statement holds for $\mathcal{M}^m_{\mu,E}$. Moreover, if the conditions above are satisfied for both $\mathcal{M}^e_{\mu_0,L_1(\mu_0)}$ and $\mathcal{M}^m_{\mu_0,L_1(\mu_0)}$, then the regularized component $\mathcal{M}^{e\#m}_{\mu,E} \equiv \mathbb{R}P^3 \# \mathbb{R}P^3$ admits a $3-2-3$ foliation whose binding orbits are P_3^e, P_3^m and P_2, for every (μ,E) sufficiently close to $(\mu_0,L_1(\mu_0))$, with $E > L_1(\mu)$.

Notice that if $\mathcal{M}^e_{\mu_0,L_1(\mu_0)}$ is dynamically convex, i.e. all of its contractible periodic orbits have index ≥ 3, then the conclusions in Theorem 1.10 hold for $\mathcal{M}^e_{\mu,E}$ for every (μ,E), sufficiently close to $(\mu_0,L_1(\mu_0))$, with $E > L_1(\mu)$. Furthermore, the proposition below shows that Theorem 1.10 also holds if $\mathcal{M}^e_{\mu_0,L_1(\mu_0)}$ is strictly convex.

Proposition 1.11. *Let $0 < \mu < 1$. If the regularized singular subset $\mathcal{M}^e_{\mu,L_1(\mu)}$ is strictly convex, then it is dynamically convex, i.e. the index of every contractible periodic orbit in $\mathcal{M}^e_{\mu,L_1(\mu)}$ is at least 3. A similar statement holds for $\mathcal{M}^m_{\mu,L_1(\mu)}$.*

This proposition is essentially proved in [36] and [54]. Indeed, consider any contractible periodic orbit P in $\mathcal{M}^e_{\mu,L_1(\mu)}$, and let $\tilde{P}$ denote a lift of P to $\widetilde{\mathcal{M}}^e_{\mu,L_1(\mu)} \subset \mathbb{R}^2 \times (\mathbb{R} \times \mathbb{R}/2\pi\mathbb{Z})$. The index of $\tilde{P}$ then depends only on the Hamiltonian near $\tilde{P}$. One may change the Hamiltonian away from $\tilde{P}$ so that $\tilde{P}$ lies in a strictly convex regular hypersurface, see [26]. Hence, its index is at least 3.

Our main result in the circular planar restricted three-body problem asserts that for every μ sufficiently close to $1/2$ and every energy slightly above the first critical value, the regularized dynamics on $\mathcal{M}^{e\#m}_{\mu,E}$ admits a $3-2-3$ foliation. Recall that the first critical value of H_μ for $\mu = 1/2$ is $L_1(1/2) = -2$.

Theorem 1.12. *The regularized subsets $\mathcal{M}^e_{1/2,E}$ and $\mathcal{M}^m_{1/2,E}$ are strictly convex for every $E \leq -2 = L_1(1/2)$.*

The proof of Theorem 1.12 involves a generalization of the results in [64] for magnetic-mechanical Hamiltonians. Checking the positivity of the sectional curvatures of $\mathcal{M}^e_{1/2,E}$ and $\mathcal{M}^m_{1/2,E}$, for $E \leq -2$, is reduced to checking the positivity of a certain function defined in the Hill region of the regularized critical subset $\widetilde{\mathcal{M}}^e_{1/2,-2}$, and a monotonicity argument that allows passing the curvature estimates from the critical value to lower energies. Finally, strict convexity follows from a simple local-to-global argument.

We are ready to state our main application of Theorem 1.10.

Theorem 1.13. *Let α, J and $\mathcal{S} = \partial\mathcal{M}^e_{\mu,E} = \partial\mathcal{M}^m_{\mu,E}$ be defined for (μ,E) sufficiently close to $(1/2,-2)$, with $E > L_1(\mu)$, as in Theorem 1.5. Let $P_3^e \subset \mathcal{M}^e_{\mu,E}$ be the continuous family of retrograde orbits given in Theorem 1.7 for (μ,E) sufficiently close to $(1/2,-2)$. The following statements hold for every (μ,E) sufficiently close to $(1/2,-2)$, with $E > L_1(\mu)$:*

(i) *The index-2 Lyapunov orbit $P_2 \subset \mathcal{M}^{e\#m}_{\mu,E}$ is the unique contractible periodic orbit with index ≤ 2. In particular, $\mathcal{M}^{e\#m}_{\mu,E}$ is weakly convex.*

(ii) *The regularized Hamiltonian flow on* $\mathcal{M}^{e\#m}_{\mu,E} \equiv \mathbb{R}P^3\#\mathbb{R}P^3$ *admits a* $3-2-3$ *foliation whose binding orbits are the retrograde orbits* P^e_3 *and* P^m_3*, and the Lyapunov orbit* P_2 *around the first Lagrange point* $l_1(\mu)$*.*
(iii) *Each chamber* $\mathcal{M}^e_{\mu,E}$ *or* $\mathcal{M}^m_{\mu,E}$ *admits infinitely many periodic orbits and infinitely many homoclinic orbits to the Lyapunov orbit near* $l_1(\mu)$*. Moreover, if the stable and unstable manifolds of the Lyapunov orbit do not coincide, then the topological entropy of the flow on* $\mathcal{M}^{e\#m}_{\mu,E}$ *is positive.*

The $3-2-3$ foliation in Theorem 1.13 follows from Theorems 1.10 and 1.12 and Proposition 1.11. We later explain how the foliation implies periodic orbits and homoclinic orbits to the Lyapunov orbit.

Remark 1.14. *For small mass ratios, homoclinic orbits to the Lyapunov orbit were studied in the regularized component* $\mathcal{M}^e_{\mu,E}$ *by McGehee* [56] *exploiting the integrability of the Rotating Kepler Problem. In fact, it is simple to check the linking and the index conditions that imply* $\mathcal{P}' = \emptyset$ *in Theorem 1.3 and thus* $\mathcal{M}^e_{\mu,E}$ *admits a* $2-3$ *foliation for* $\mu > 0$ *sufficiently small and* E *slightly above* $L_1(\mu)$*. McGehee's construction is relatively simpler than finding a* $2-3$ *foliation but somehow deals with the existence of a family of disks bounded by the retrograde orbit that are transverse to the flow.*

Remark 1.15. *Numerical computations suggest that the critical components* $\mathcal{M}^e_{\mu,L_1(\mu)}$ *and* $\mathcal{M}^m_{\mu,L_1(\mu)}$ *are strictly convex in elliptic coordinates for a large open interval around* $\mu = 1/2$*. For* $\mu > 0$ *sufficiently small, although* $\mathcal{M}^e_{\mu,L_1(\mu)}$ *is not strictly convex, it is dynamically convex. See* [48] *for a computer-assisted proof of convexity in a certain range of mass ratios and energies below* $L_1(\mu)$*. Finally, the critical subsets* $\mathcal{M}^e_{\mu,L_1(\mu)}$ *and* $\mathcal{M}^m_{\mu,L_1(\mu)}$ *are expected to be dynamically convex for every mass ratio and thus* $3-2-3$ *foliations should exist slightly above the first Lagrange value for every mass ratio.*

1.4. **Birkhoff's retrograde orbit conjecture.**

In [6], Birkhoff raised the question of whether the double cover of the retrograde orbit P^e_3 bounds a disk-like global surface of section for the flow on the regularized components $\mathcal{M}^e_{\mu,E}, \mathcal{M}^m_{\mu,E} \equiv \mathbb{R}P^3$, $E < L_1(\mu)$. This question was motivated by the difficulty of proving the existence of a direct orbit around the primaries. The convexity estimates given in Theorem 1.12 imply that for all mass ratios sufficiently close to $1/2$ and all energies below the first Lagrange value, the regularized components $\mathcal{M}^e_{\mu,E}$ and $\mathcal{M}^m_{\mu,E}$ are dynamically convex, i.e., all of its contractible periodic orbits have Conley-Zehnder index at least 3. This positively answers Birkhoff's conjecture for such mass ratios and energies.

Theorem 1.16. *There exists* $\epsilon_0 > 0$ *such that for every* $|\mu - 1/2| < \epsilon_0$ *and* $E < L_1(\mu)$*, the following holds:*

(i) *The* $\mathbb{R}P^3$*-components* $\mathcal{M}^e_{\mu,E}$ *and* $\mathcal{M}^m_{\mu,E}$ *are dynamically convex, i.e., every contractible periodic orbit has index at least* 3*.*
(ii) *Every retrograde orbit* $P^e_3 \subset \mathcal{M}^e_{\mu,E} \equiv \mathbb{R}P^3$ *binds a rational open book decomposition whose pages are disk-like global surfaces of section. More generally, the same holds for every periodic orbit* $P \subset \mathcal{M}^e_{\mu,E}$ *which is transversely isotopic to a Hopf fiber. A similar statement holds for* $\mathcal{M}^m_{\mu,E}$*.*
(iii) *Let* $P' \subset \mathcal{M}^e_{\mu,E}$ *be the simple periodic orbit corresponding to a fixed point of the first return map associated to the global surface of section bounded by* P *as in (ii). Then the Hopf link* $P \cup P'$ *bounds an annulus-like global surface of section. A similar statement holds for* $\mathcal{M}^m_{\mu,E}$*.*

1.5. **Related works.**

The theory of pseudo-holomorphic curves in symplectizations was initiated by Hofer [29] and developed by Hofer, Wysocki, and Zehnder [33, 34, 35]. The first breakthrough concerning finite energy foliations was motivated by the dynamics on strictly convex hypersurfaces in $\mathbb{R}^4$.

Theorem 1.17 (Hofer-Wysocki-Zehnder [36])**.** *Let* $\alpha = f\alpha_0$ *be a dynamically convex contact form on the tight three-sphere* (S^3, ξ_0)*. Then* α *admits an index-*3 *periodic orbit binding an open* *book decomposition whose pages are disk-like global surfaces of section. Such an open book is the projection of a finite energy foliation in the symplectization.*

The existence of finite energy foliations projecting to open book decompositions, all of whose pages are global surfaces of section, was further studied in [31, 32, 38, 39, 40, 41, 42], see the surveys [7, 30, 63].

A fundamental result by Hofer, Wysocki, and Zehnder establishes more general finite energy foliations for generic star-shaped hypersurfaces in $\mathbb{R}^4$.

Theorem 1.18 (Hofer-Wysocki-Zehnder [37]). *Let $\alpha = f\alpha_0$ be a nondegenerate contact form on the tight three-sphere (S^3, ξ_0). Then, for J in a generic subset $\mathcal{J}_{reg}(\alpha) \subset \mathcal{J}(\alpha)$ of $d\alpha$-compatible almost complex structures, there exists a finite energy foliation by J-holomorphic curves whose projection is a transverse foliation. The binding is formed by simple periodic orbits with self-linking number -1 and index 1, 2, or 3. Each regular leaf is the projection of a J-holomorphic curve, has genus zero and satisfies one of the following conditions:*

(i) *It has one positive puncture asymptotic to an index-3 orbit and an arbitrary number of negative punctures asymptotic to index-1 orbits.*
(ii) *It has one positive puncture asymptotic to an index-3 orbit, one negative puncture asymptotic to an index-2 orbit, and an arbitrary number of negative punctures asymptotic to index-1 orbits.*
(iii) *It has one positive puncture asymptotic to an index-2 orbit and an arbitrary number of negative punctures asymptotic to index-1 orbits.*

Wendl [69, 70, 71, 72] developed further the theory of finite energy curves, constructed finite energy foliations for overtwisted contact manifolds, and established conditions for a finite energy curve to be Fredholm regular. The results in [72] are useful in proving Theorem 1.3.

Fish and Siefring [21] studied connected sums of finite energy foliations. The existence of finite energy foliations near critical energy surfaces was considered in [15]. Colin, Dehornoy, and Rechtmann [8] combined finite energy curves and Fried theory of asymptotic cycles to obtain the so-called broken books, leading to deep results in Reeb dynamics regarding periodic orbits and global surfaces of sections [9, 13, 14].

At the beginning of the twentieth century, Poincaré [60] and Birkhoff [6] made important contributions to the circular planar restricted three-body problem concerning the existence of periodic orbits and global surfaces of section. Poincaré proved the existence of annulus-like global surfaces of section on the earth-side of the regularized energy surface for energies below the first Lagrange value and small mass ratios. His argument was perturbative, strongly relying on the integrability of the Rotating Kepler Problem. Birkhoff used the shooting method to find retrograde orbits for every mass ratio and energy below the first Lagrange value. Conley [10, 11, 12] and McGehee [56] studied periodic orbits and homoclinic orbits to the Lyapunov orbits for larger energies.

The employment of holomorphic curves methods in the circular planar restricted three-body problem started in [1] with the important observation that the energy surfaces are contact-type. In [2], the authors discussed the existence of global surfaces of section near the moon for small mass ratios by checking the strict convexity of the regularized sphere-like component of the energy surface. Moreno and van Koert [57] found global hypersurfaces of section in the spatial restricted three-body problem for energies up to slightly above the first Lagrange value. In [44], the authors showed the existence of global surfaces of section in the spatial isosceles three-body problem. Several other concrete systems admitting finite energy foliations, including classical dynamical systems, were found in [15, 16, 18, 49, 50, 64, 65]. A method to create finite energy foliations with specific binding orbits was introduced in [17] for weakly convex Reeb flows on the tight three-sphere. This approach was utilized in [18] to produce weakly convex foliations in the Hénon-Heiles system for energies slightly above the critical value.

We recommend the book by Frauenfelder and van Koert [24] for a concise introduction to symplectic methods in celestial mechanics, with emphasis in the circular planar restricted three-body problem.

Acknowledgement. LL is partially supported by the National Natural Science Foundation of China (Grant number: 12401238), the Natural Science Foundation of Shandong Province, China (Grant number: ZR2024QA188). LL thanks the support of the School of Mathematics at Shandong University. PS thanks Umberto Hryniewicz for several conversations on the analysis of J-holomorphic curves in symplectizations. PS was partially supported by the National Natural Science Foundation of China (grant number W2431007). LL and PS thanks the support of the Shenzhen International Center for Mathematics - SUSTech.

2. Preliminaries

2.1. **Reeb flows and pseudo-holomorphic curves.** The pair $(S^3, \xi_0 = \ker \alpha_0)$ is a contact manifold called the tight three-sphere, where α_0 is the Liouville form restricted to S^3. For every smooth function $f : S^3 \to \mathbb{R}^+$, $\alpha := f\alpha_0$ is a contact form on (S^3, ξ_0), and its Reeb vector field R is determined by $d\alpha(R, \cdot) = 0$ and $\alpha(R) = 1$. The flow of R, denoted $\psi_t, t \in \mathbb{R}$, preserves α and thus preserves ξ_0. We say that a periodic orbit $P = (x, T)$ is nondegenerate if the linear map $D\psi_T(x(0)) : \xi_0|_{x(0)} \to \xi_0|_{x(0)}$ does not have 1 as an eigenvalue. We say that α is nondegenerate if all of its periodic orbits are nondegenerate. The iterates of P are denoted $P^k = (x, kT)$, for every $k \in \mathbb{N}$.

Given relatively prime integers $p \geq q \geq 1$, we consider the lens space $L(p, q) = S^3/\mathbb{Z}_p$ as before. Since the contact form α_0 on S^3 is $\mathbb{Z}_p$-invariant, it descends to a contact form on $L(p, q)$, also denoted α_0. The contact structure $\xi_0 := \ker \alpha_0$ is called the universally tight contact structure on $L(p, q)$.

A knot $K \subset L(p, q)$ is called p-unknotted if there exists an immersion $u : \mathbb{D} \to L(p, q)$ so that $u|_{\mathbb{D} \setminus \partial\mathbb{D}} : \mathbb{D} \setminus \partial\mathbb{D} \to L(p, q) \setminus K$ is an embedding and $u|_{\partial\mathbb{D}} : \partial\mathbb{D} \to K$ is a p-covering map. The disk u is called a p-disk for K. Let $K \subset L(p, q)$ be p-unknotted and transverse to ξ_0. Take a p-disk u for K and a small non-vanishing section Y of $u^*\xi_0$. Use Y and an exponential map to push K to a knot K' that is disjoint from K, close to a p-cover of K, and transverse to u. The (rational) self-linking number of K is defined as the normalized algebraic intersection number between K' and u, i.e. $\mathrm{sl}(K) := \frac{1}{p^2} K' \cdot u$. Here, K is oriented by α_0, K' inherits the orientation of K, u is oriented by K and $L(p, q)$ is oriented by $\alpha_0 \wedge d\alpha_0 > 0$. As an example, the knot $K := \pi_{p,q}(S^1 \times 0)$ is p-unknotted and transverse to ξ_0. In this case, a p-disk for K is given by $\pi_{p,q} \circ u$, where $u(z) = (z, \sqrt{1 - |z|^2}) \in S^3, \forall z \in \mathbb{D}$. One readily checks that a knot K' as above satisfies $K' \cdot u = -p$, and thus $\mathrm{sl}(K) = -\frac{1}{p}$.

Let $B_i \subset L(p, q), i = 1, \ldots, l$, be mutually disjoint regular open three-balls, and let $\mathcal{M} := L(p, q) \setminus \cup_{i=1}^{l} B_i$. Then $\partial\mathcal{M}$ is the union of l regular two-spheres $\mathcal{S}_i := \partial B_i$. The restriction of α_0 and ξ_0 to $\mathcal{M}$ is still denoted α_0 and ξ_0, respectively. We only require $\partial\mathcal{M}$ to be C^1.

Let J be an almost complex structure on $\mathbb{R} \times \mathcal{M}$ so that $J \cdot \partial_a = R$ and $J(\xi_0) = \xi_0$, where $d\alpha(\cdot, J, \cdot)$ is an inner product on ξ_0. Here, a is the $\mathbb{R}$-coordinate, and R and ξ are regarded as $\mathbb{R}$-invariant objects on $\mathbb{R} \times \mathcal{M}$. The space of such J's is denoted by $\mathcal{J}(\alpha)$. Let (Σ, j) be a connected Riemann surface (possibly with boundary), and let $\Gamma \subset \Sigma \setminus \partial\Sigma$ be a finite set. Let $\dot{\Sigma} := \Sigma \setminus \Gamma$, and let $J \in \mathcal{J}(\alpha)$. A map $\tilde{u} = (a, u) : \dot{\Sigma} \to \mathbb{R} \times \mathcal{M}$ is called a finite energy J-holomorphic curve if it satisfies the non-linear Cauchy-Riemann equation $\bar{\partial}_J \tilde{u} = d\tilde{u} + J(\tilde{u}) \circ d\tilde{u} \circ j = 0$, and has finite Hofer energy $0 < E(\tilde{u}) := \sup_{\phi \in \mathcal{T}} \int_{\dot{\Sigma}} \tilde{u}^* d(\phi(a)\alpha) < +\infty$, where $\mathcal{T} := \{\phi : \mathbb{R} \to [0, 1],\ \phi' \geq 0\}$.

We consider totally real boundary conditions. Let $\gamma_1, \ldots, \gamma_{m_0}$ be the components of $\partial\Sigma$. We assume that for each $j = 1, \ldots, m_0$, there exists a totally real surface $L_j \subset \{0\} \times \mathcal{M}$, that is $TL_j \oplus JTL_j = T(\mathbb{R} \times \mathcal{M})$ along L_j, so that $\tilde{u}(\gamma_j) \subset L_j$. We denote by $l_j \to \gamma_j$ the line bundle of $u^*\xi_0|_{\gamma_j}$ given by $l_j(z) = \xi_0|_{u(z)} \cap T_{u(z)} L_j$ for every $z \in \gamma_j$. Later we shall consider the particular case where $L \subset \{0\} \times (\mathcal{M} \setminus \partial\mathcal{M})$ is a totally real surface that transversely intersects the contact structure and $\tilde{u} = (a, u) : \mathbb{D} \to \mathbb{R} \times \mathcal{M}$ is a J-holomorphic disk satisfying $\tilde{u}(\partial\mathbb{D}) \subset L$ and $\partial_{\vec{n}} a > 0$, where $\vec{n}$ is the outer normal vector along $\partial\mathbb{D}$.

Each non-removable puncture $z_0 \in \Gamma$ of $\tilde{u}$ has a sign $\epsilon(z) \in \{-1, +1\}$ so that $a(z) \to \epsilon(z_0)\infty$ as $z \to z_0$. Furthermore, for suitable polar coordinates $s + it \in [0, +\infty) \times \mathbb{R}/\mathbb{Z}$ on a punctured neighborhood of z_0 the following holds: given a sequence $s_n \to +\infty$, there exists a subsequence also denoted s_n and a periodic orbit $P=(x,T)$ so that $u(s_n,\cdot)\to x(\epsilon(z_0)T\cdot)$ in C^∞ uniformly in t as $s\to+\infty$. The periodic orbit P is called an asymptotic limit of $\tilde u$ at z_0, and if P is nondegenerate, then P is the unique asymptotic limit of $\tilde u$ at z_0, and much can be said about the asymptotic behavior of $\tilde u$ as it approaches P. If $\epsilon(z_0)=+1$, we say that z_0 is a positive puncture. Otherwise, we say z_0 is a negative puncture. The signs of the punctures induce the splitting $\Gamma=\Gamma^+\cup\Gamma^-$.

2.2. **The asymptotic operator and the Conley-Zehnder index.** Let $P=(x,T)$ be a periodic orbit of α and let $x_T:=x(T\cdot):\mathbb{R}/\mathbb{Z}\to\mathcal{M}$. Let $J\in\mathcal{J}(\alpha)$. The unbounded self-adjoint operator $A_P:W^{1,2}(\mathbb{R}/\mathbb{Z},x_T^*\xi_0)\to L^2(\mathbb{R}/\mathbb{Z},x_T^*\xi_0)$, defined by $A_P\cdot\eta:=-J\cdot\mathcal{L}_{\dot x_T}\eta$, is called the asymptotic operator of P. Here, $\mathcal{L}_{\dot x_T}\eta$ is the Lie derivative of η along x_T. The spectrum $\sigma(A_P)$ of A_P consists of countably many real eigenvalues accumulating precisely at $\pm\infty$. An eigenvector $e:\mathbb{R}/\mathbb{Z}\to\mathbb{R}^2$ of $\lambda\in\sigma(A_P)$ is smooth and never vanishes. Hence, for a fixed trivialization $\Psi:x_T^*\xi_0\to\mathbb{R}/\mathbb{Z}\times\mathbb{R}^2$, e has a well-defined winding number $\text{wind}(\lambda)$, depending only on λ and the homotopy class of Ψ. We omit the dependence on Ψ in the notation. The function $\sigma(A_P)\ni\lambda\mapsto\text{wind}(\lambda)\in\mathbb{Z}$ is monotone increasing, and given $k\in\mathbb{Z}$, there exist precisely 2 eigenvalues (counting multiplicities) of A_P with winding number k. It can be directly checked that P is nondegenerate if and only if $0\notin\sigma(A_P)$. Fix $\delta\in\mathbb{R}$, and let

$$\text{wind}^{<\delta}(A_P):=\max\{\text{wind}(\lambda):\sigma(A_P)\ni\lambda<\delta\},$$
$$\text{wind}^{\geq\delta}(A_P):=\min\{\text{wind}(\lambda):\sigma(A_P)\ni\lambda\geq\delta\}.$$

The weighted Conley-Zehnder index of P is defined as

$$\mu^\delta(P):=\text{wind}^{<\delta}(A_P)+\text{wind}^{\geq\delta}(A_P).$$

The weighted index $\mu^\delta(P)$ depends on J and the homotopy class of Ψ. If $\delta=0$, then $\mu(P):=\mu^0(P)$ depends only on the homotopy class of Ψ. Since the parity of $\mu(P)$ does not depend on Ψ, there exists a natural splitting $\Gamma=\Gamma_{\text{even}}\cup\Gamma_{\text{odd}}$.

Having a geometric definition of $\mu(P)$ is convenient. In the frame induced by Ψ, the linearized flow along $x(t)$ determines a path of 2×2 symplectic matrices $t\mapsto\Phi(t),t\in[0,T]$. Given $v_0\in\mathbb{R}^2\setminus\{0\}$, let $\theta(t),t\in[0,T]$, be a continuous argument of $\Phi(t)\cdot v_0$, and let $\Delta(v_0):=(\theta(T)-\theta(0))/2\pi$. Then $I_P:=\{\Delta(v_0),0\neq v_0\in\mathbb{R}^2\}$ is an interval of length $<1/2$, and there exists $k\in\mathbb{Z}$ such that for every $\epsilon>0$ sufficiently small, either $k\in I_P-\epsilon$ or $I_P-\epsilon\subset(k,k+1)$. We then have, respectively, $\mu(P)=2k$ or $\mu(P)=2k+1$. Finally, the rotation number of P in the frame induced by Ψ is defined as $\rho(P):=\lim_{k\to\infty}\frac{1}{2k}\mu(P^k)$, where P^k is the k-th iterate of P. It is immediate to check that if $\mu(P)=2$, then $\rho(P)=1$. Moreover, $\mu(P)\geq 3$ if and only if $\rho(P)>1$.

2.3. **Asymptotics of J-holomorphic curves.** Let $z_0\in\Gamma$ be a puncture of a J-holomorphic curve $\tilde u=(a,u):\Sigma\setminus\Gamma\to\mathbb{R}\times M$ and let $\epsilon(z_0)\in\{-1,+1\}$ be the sign of z_0. Assume that $P=(x,T)$ is an asymptotic limit of $\tilde u$ at z_0. Denote by $P_0=(x_0,T_0)$ the simple periodic orbit so that $P=P_0^k$ for some integer $k\geq 1$. Let $(\vartheta,x,y)\in\mathbb{R}/\mathbb{Z}\times B_\delta(0)$ be coordinates on a small tubular neighborhood $\mathcal{U}\subset M$ of P_0, so that $\alpha=f(d\vartheta+xdy)$, for a function $f=f(\vartheta,x,y)$ satisfying $f(\vartheta,0,0)=T_0$ and $df(\vartheta,0,0)=0$ for every ϑ. Here, $B_\delta(0)\subset\mathbb{R}^2$ is an open disk of radius $\delta>0$ centered at 0. Let $z:=(x,y)$. Consider polar coordinates $s+it\in[0,\infty)\times\mathbb{R}/\mathbb{Z}\to e^{-2\pi(s+it)}\in\mathbb{D}\setminus\{0\}$ on a punctured neighborhood of $z_0\equiv 0$ and write $\tilde u(s,t)=(a(s,t),\vartheta(s,t),z(s,t))$, whenever defined. Let A_P be the asymptotic operator at P.

Theorem 2.1 (Hofer-Wysocki-Zehnder [33], Siefring [66]). *Assume that $z_0\in\Gamma$ is a positive puncture and that $P=P_0^k$ is nondegenerate. Then $(\vartheta(s,t),z(s,t))\in\mathbb{R}/\mathbb{Z}\times B_\delta(0)$ for every s sufficiently large, and there exist $a_0,\vartheta_0\in\mathbb{R}$, a λ-eigenvector $e(t)$ of A_P, with $\lambda<0$, so that*

$$z(s,t)=e^{\lambda s}(e(t)+r(s,t)),\tag{2.1}$$

where $|r(s,t)|\to 0$ as $s\to+\infty$ uniformly in t. Moreover, $|a(s,t)-(a_0+Ts)|\to 0$ and $|\vartheta(s,t)-(kt+\vartheta_0)|\to 0$ as $s\to+\infty$ uniformly in t, where ϑ is lifted to a real-valued function.

In Theorem 2.1, λ and e are called the leading eigenvalue and the leading eigenvector of $\tilde u$ at $z_0\in\Gamma$, respectively. The leading eigenvector is determined up to a positive multiple. If z_0 is a negative puncture, then we consider polar coordinates $(s,t)\in(-\infty,0]\times\mathbb{R}/\mathbb{Z}\to e^{2\pi(s+it)}\in$

$\mathbb{D} \setminus \{0\}$ on a punctured neighborhood of $z_0 \equiv 0$ and the asymptotic formula (2.1) still holds for some positive leading eigenvalue λ and a leading eigenvector e. The asymptotic properties of $a(s,t) \to -\infty$ and $\vartheta(s,t)$ as $s \to -\infty$ are similar to the case of a positive puncture.

We call a puncture $z_0 \in \Gamma$ nondegenerate if $\tilde{u}$ has an asymptotic formula at z_0 as in Theorem 2.1 for a non-vanishing leading eigenvalue λ and a leading eigenvector e. With this definition, the puncture z_0 may be nondegenerate even if the asymptotic limit P is degenerate.

Definition 2.2. *Let $\tilde{u}_i : \Sigma \setminus \Gamma_i \to \mathbb{R} \times M$, $i = 1, 2$, be a pair of finite energy J-holomorphic curves asymptotic to the same nondegenerate periodic orbit $P = (x, T)$ at $z_i \in \Gamma_i$. Let λ_i and e_i be the leading eigenvalue and leading eigenvector of $\tilde{u}_i$ at z_i, respectively. We say that $\tilde{u}_1$ and $\tilde{u}_2$ approach P through the same direction if $\lambda_1 = \lambda_2$ and $e_1 = ce_2$ for some constant $c > 0$. If $e_1 = -ce_2$ for some $c > 0$ then we say that $\tilde{u}_1$ and $\tilde{u}_2$ approach P through opposite directions.*

2.4. **Uniqueness of J-holomorphic planes and cylinders.** The following uniqueness results on J-holomorphic curves follows from Siefring's intersection theory [66, 67].

Theorem 2.3. *The following uniqueness statements hold:*

(i) *If $\tilde{u} = (a, u) : \mathbb{C} \to \mathbb{R} \times \mathcal{M}$ is an embedded J-holomorphic plane asymptotic to $P_{2,i}$, then up to parametrization and $\mathbb{R}$-translation, $\tilde{u}$ coincides with one of the J-holomorphic planes projecting to the hemispheres of $\mathcal{S}_i \setminus P_{2,i}$.*

(ii) *If $\tilde{v}_1 = (b, v), \tilde{v}_2 = (b_2, v_2) : \mathbb{R} \times \mathbb{R}/\mathbb{Z} \to \mathbb{R} \times (\mathcal{M} \setminus \partial\mathcal{M})$ are embedded J-holomorphic cylinders, not intersecting $\mathbb{R} \times P_3$, with a positive end at P_3^p whose leading eigenvalue has winding number 1, and a negative end at $P_{2,i}$, then up to parametrization and $\mathbb{R}$-translation, $\tilde{v}_1$ coincides with $\tilde{v}_2$.*

Proof. The proof of (i) is essentially contained in [15, Proposition C-3]. However, the proof of (ii) needs to be adapted to the current situation since the index of P_3^p might be greater than 3. In that case, the leading eigenvalue of $\tilde{v}_i$ at the positive puncture has winding number 1 and thus does not coincide with the winding of the largest negative eigenvalue of $A_{P_3^p}$.

Let $\tilde{v}_1, \tilde{v}_2$ be as in Theorem (ii). Siefring [67] introduced the generalized intersection number $[\tilde{v}_1] * [\tilde{v}_2]$, which is invariant under homotopies that keep the same asymptotic limits. This number counts actual intersections between the curves as well as intersections at infinity, i.e., intersections related to the respective punctures of $\tilde{v}_1$ and $\tilde{v}_2$ whose asymptotic limits are covers of the same simple periodic orbit. In particular, this number includes hidden intersections at the punctures, which correspond to tangencies at infinity or appear only when the curves are suitably perturbed. Since $\tilde{v}_1$ and $\tilde{v}_2$ have the same asymptotic limits at their respective positive and negative punctures, and since these curves do not intersect the trivial cylinders $\mathbb{R} \times P_3$ and $\mathbb{R} \times P_{2,i}$, Theorem 5.8 from [67] gives

$$[\tilde{v}_1] * [\tilde{v}_2] = pd_0^+ + d_0^-,$$

where $d_0^+ \geq 0$ is the difference between the winding number $\alpha_+ := d + 1 \geq 1$ of the largest negative eigenvalue of $A_{P_3^p}$ and the winding number of the leading eigenvalue at the positive puncture of $\tilde{v}_1$, which is equal to 1. In the same way, $d_0^- \geq 0$ is the difference between the winding number of the leading eigenvalue at the negative puncture, which is equal to 1, and the winding number α_- of the smallest positive eigenvalue of $A_{P_3^p}$, also equal to 1. Indeed, at the negative puncture, if the leading eigenvalue does not have winding number 1, then its projection to $\mathcal{M}$ must wind around $P_{2,i}$ with winding number ≥ 2, forcing intersections with $\partial\mathcal{M}$, a contradiction. Hence we obtain $d_0^+ = d$ and $d_0^- = 0$, which implies that

$$[\tilde{v}_1] * [\tilde{v}_2] = pd. \tag{2.2}$$

This value accounts for the hidden intersections at P_3^p that arise from the positive puncture after suitably perturbing $\tilde{v}_1$ and $\tilde{v}_2$. Since the asymptotic limit is the p-cover of a simple periodic orbit, the number of these potential intersections is a multiple of p.

The generalized intersection number $[\tilde{v}_1] * [\tilde{v}_2]$ can be better expressed once we have more information on the relative asymptotic behavior between $\tilde{v}_1$ and $\tilde{v}_2$ at both positive and negative punctures. Assume by contradiction that $\tilde{v}_1$ does not coincide with $\tilde{v}_2$, i.e., $\tilde{v}_1$ is not obtained from $\tilde{v}_2$ by reparametrization and $\mathbb{R}$-translation. In particular, $v_1(\mathbb{R}\times\mathbb{R}/\mathbb{Z}) \neq v_2(\mathbb{R}\times\mathbb{R}/\mathbb{Z})$ and one may consider the non-trivial difference between these curves near the ends. It is then proved in [67, Theorem 4.4] that

$$[\tilde{v}_1] * [\tilde{v}_2] = \mathrm{int}(\tilde{v}_1, \tilde{v}_2) + \delta_\infty(\tilde{v}_1, \tilde{v}_2), \tag{2.3}$$

where $\mathrm{int}(\tilde{v}_1, \tilde{v}_2) \geq 0$ is the algebraic intersection number of $\tilde{v}_1$ and $\tilde{v}_2$ and $\delta_\infty(\tilde{v}_1, \tilde{v}_2) \geq 0$ is the total asymptotic intersection index of $\tilde{v}_1$ and $\tilde{v}_2$, i.e.,

$$\delta_\infty(\tilde{v}_1, \tilde{v}_2) = \delta_\infty^+(\tilde{v}_1, \tilde{v}_2) + \delta_\infty^-(\tilde{v}_1, \tilde{v}_2).$$

Here, $\delta_\infty^+(\tilde{v}_1, \tilde{v}_2)$ and $\delta_\infty^-(\tilde{v}_1, \tilde{v}_2)$ are the asymptotic intersection indices of $\tilde{v}_1$ and $\tilde{v}_2$ at the positive and negative punctures, respectively, given by

$$\begin{aligned}\delta_\infty^+(\tilde{v}_1, \tilde{v}_2) &= i_\infty^+(\tilde{v}_1, \tilde{v}_2) + p\alpha_+,\\ \delta_\infty^-(\tilde{v}_1, \tilde{v}_2) &= i_\infty^-(\tilde{v}_1, \tilde{v}_2) - \alpha_-,\end{aligned}$$

see Lemma 3.20 and equation (3-32) in [67]. Notice that the sign convention in [67] is slightly different from the one in this paper. Indeed, in [67], the winding number at a negative puncture has an opposite sign. Here,

$$i_\infty^\pm(\tilde{v}_1, \tilde{v}_2) = \mp\mathrm{wind}_{\mathrm{rel}}^\pm(\tilde{v}_1, \tilde{v}_2)$$

is the adjusted winding number between the difference of $\tilde{v}_1$ and $\tilde{v}_2$ on suitable coordinates, see [66]. In the case of positive punctures, where the curves p-cover P_3, since the leading eigenvectors of both curves have winding number 1 and the space of such eigenvectors is two-dimensional, the winding number of the difference between $\tilde{v}_1$ and $\tilde{v}_2$ in suitable coordinates is ≤ 1. The fact that the asymptotic limit is the p-cover of P_3, this number is multiplied by p. Therefore,

$$i_\infty^+(\tilde{v}_1, \tilde{v}_2) = -\mathrm{wind}_{\mathrm{rel}}^+(\tilde{v}_1, \tilde{v}_2) \geq -p,$$

see Corollary 3.21 in [67]. This implies that

$$\delta_\infty^+(\tilde{v}_1, \tilde{v}_2) \geq -p + p(d+1) = pd.$$

Now, since both leading eigenvalues of $\tilde{v}_1$ and $\tilde{v}_2$ at their negative punctures coincide and have winding number 1, and since the space of such eigenvectors is one-dimensional, we conclude from Siefring's formula [66] for the difference between $\tilde{v}_1$ and $\tilde{v}_2$ near the negative punctures that

$$i_\infty^-(\tilde{v}_1, \tilde{v}_2) = \mathrm{wind}_{\mathrm{rel}}^-(\tilde{v}_1, \tilde{v}_2) \geq 2.$$

This implies that

$$\delta_\infty^-(\tilde{v}_1, \tilde{v}_2) \geq 2 - \alpha_- = 2 - 1 = 1.$$

The above estimates for $\delta_\infty^+(\tilde{v}_1, \tilde{v}_2)$ and $\delta_\infty^-(\tilde{v}_1, \tilde{v}_2)$ give

$$\delta_\infty(\tilde{v}_1, \tilde{v}_2) = \delta_\infty^+(\tilde{v}_1, \tilde{v}_2) + \delta_\infty^-(\tilde{v}_1, \tilde{v}_2) \geq pd + 1,$$

and thus (2.3) implies $[\tilde{v}_1] * [\tilde{v}_2] \geq pd + 1$, contradicting (2.2). We conclude that $\tilde{v}_1$ coincides with $\tilde{v}_2$ up to reparametrization and $\mathbb{R}$-translation and this finishes the proof of Theorem 2.3-(ii). □

2.5. **Automatic Transversality.** Fix a compact connected Riemann surface (Σ, j) possibly with non-empty boundary, and let $\Gamma \subset \Sigma \setminus \partial\Sigma$ be a finite set. Assume that all punctures of a J-holomorphic curve $\tilde{u} = (a, u) : \Sigma\setminus\Gamma \to \mathbb{R}\times\mathcal{M}$ are nondegenerate and denote by P_z the asymptotic limit of $\tilde{u}$ at $z \in \Gamma$. Assume that the $d\alpha$-area of $\tilde{u}$ is positive. Fix a symplectic trivialization Ψ of $u^*\xi$, which induces a homotopy class of symplectic trivializations of ξ along the asymptotic limits $P_z, z \in \Gamma$. Denote by $\mu(P_z), z \in \Gamma$, the index of P_z induced by Ψ. Let λ_z be the leading eigenvalue of $\tilde{u}$ at z. Recall that $\lambda_z < 0$ if z is a positive puncture, and $\lambda_z > 0$, otherwise. Let δ be a collection of real numbers $\delta_z, z \in \Gamma$, called weights, so that $\lambda_z < \delta_z \leq 0$ if z is a positive puncture and $0 \leq \delta_z < \lambda_z$, otherwise. In the following, we shall assume that δ_z is not an eigenvalue of A_{P_z}. Later, we also assume that no eigenvalue of A_{P_z} exists between λ_z and δ_z. Assume that each boundary component $\gamma_j \subset \partial\Sigma$ is mapped under $\tilde{u}$ into a totally real surface $L_j \subset \{0\}\times M$ and let

$l_j \to \gamma_j$ be the line bundle of $u^*\xi|_{\gamma_j}$ given by $l_j(z) = \xi|_{u(z)} \cap T_{u(z)}L_j, \forall z \in \gamma_j$ and $j = 1, \cdots, m_0$. The δ-weighted Conley-Zehnder index and the δ-weighted Fredholm index of $\tilde{u}$ are defined as

$$\begin{aligned} \mu^\delta(\tilde{u}) &:= \sum_{j=1}^{m_0} \mu(u^*\xi|_{\gamma_j}, l_j) + \sum_{z\in\Gamma^+} \mu^{\delta_z}(P_z) - \sum_{z\in\Gamma^-} \mu^{\delta_z}(P_z), \\ \mathrm{ind}^\delta(\tilde{u}) &:= \mu^\delta(\tilde{u}) - \chi(\dot{\Sigma}), \end{aligned} \tag{2.4}$$

where $\chi(\dot{\Sigma}) = \chi(\Sigma) - \#\Gamma$ is the Euler characteristic of $\dot{\Sigma}$. The first term in the definition of $\mu^\delta(\tilde{u})$ consists of Maslov indices of $l_j \subset u^*\xi|_{\gamma_j}$ in the frame Ψ, and m_0 is the number of components of $\partial\Sigma$. If $\delta_z = 0, \forall z \in \Gamma$, then $\mathrm{ind}^\delta(\tilde{u})$ is denoted by $\mathrm{ind}(\tilde{u})$ and called the index of $\tilde{u}$. Since $\mu^{\delta_z}(P_z) \leq \mu(P_z), \forall z \in \Gamma^+$, and $\mu^{\delta_z}(P_z) \geq \mu(P_z), \forall z \in \Gamma^-$, the inequalities $\mu^\delta(\tilde{u}) \leq \mu(\tilde{u})$ and $\mathrm{ind}^\delta(\tilde{u}) \leq \mathrm{ind}(\tilde{u})$ always hold.

Recall that the $d\lambda$-area $\int_{\dot{\Sigma}} u^*d\lambda$ of $\tilde{u}$ is always non-negative and vanishes if and only if $\pi \circ du \equiv 0$, where $\pi : TM \to \xi$ is the projection along the Reeb vector field. We keep assuming that $\int_{\dot{\Sigma}} u^*d\lambda > 0$. In this case, the leading eigenvalue and leading eigenvector are well-defined at each puncture. Since $\dot{\Sigma}$ is connected, Theorem 2.1 implies that $\pi \circ du$ does not vanish near the punctures. Since $\pi \circ du$ satisfies a Cauchy-Riemann equation, each zero is isolated and has a positive local degree. Hence, $\pi \circ du$ has finitely many zeros, and the sum of their local degree is denoted by $\mathrm{wind}_\pi(\tilde{u})$. Each zero of $\pi \circ du$ lying in $\partial\Sigma$ contributes with half of its local degree. Hence $\mathrm{wind}_\pi(\tilde{u})$ is a half-integer. According to Theorem 2.1, each puncture $z \in \Gamma$ admits a leading eigenvalue and a leading eigenvector. The winding number of the leading eigenvector in the frame Ψ is denoted by $\mathrm{wind}_\infty(z)$. It is proved in [34] that $\mathrm{wind}_\pi(\tilde{u}) = \sum_{z\in\Gamma^+} \mathrm{wind}_\infty(z) - \sum_{z\in\Gamma^-} \mathrm{wind}_\infty(z) - \chi(\dot{\Sigma})$ provided $\partial\Sigma = \emptyset$. The theorem below follows from Wendl's results in [70] and the definitions above.

Theorem 2.4 (Wendl [70])**.** *The following inequalities hold*

$$0 \leq 2\mathrm{wind}_\pi(\tilde{u}) \leq \mathrm{ind}^\delta(\tilde{u}) - 2 + 2g + \#\Gamma^\delta_{\mathrm{even}} + m_0, \tag{2.5}$$

where g is the genus of Σ and $\#\Gamma^\delta_{\mathrm{even}}$ is the number of punctures whose asymptotic limit has an even δ-weighted index.

Assume that the J-holomorphic curve $\tilde{u} = (a, u) : \dot{\Sigma} \to \mathbb{R} \times \mathcal{M}$ is embedded. Then $\tilde{u}^*T(\mathbb{R} \times \mathcal{M})$ splits as $T_{\tilde{u}} \oplus N_{\tilde{u}}$, where the fiber of $T_{\tilde{u}}$ at $z \in \dot{\Sigma}$ is $d\tilde{u}(T_z\dot{\Sigma})$ and $N_{\tilde{u}}$ is a complex line bundle complementary to $T_{\tilde{u}}$. From the asymptotic behavior of $\tilde{u}$, see Theorem 2.1, we may assume that $N_{\tilde{u}}$ coincides with $\tilde{u}^*\xi$ near the punctures. We also assume that $N_{\tilde{u}}$ coincides with $u^*\xi|_{\partial\Sigma}$ along $\partial\Sigma$.

The δ-weighted normal first Chern number $c_N^\delta(\tilde{u})$ of $\tilde{u}$ is defined as the half-integer

$$2c_N^\delta(\tilde{u}) := \mathrm{ind}^\delta(\tilde{u}) - 2 + 2g + \#\Gamma^\delta_{\mathrm{even}} + m_0, \tag{2.6}$$

where g is the genus of Σ and $\Gamma^\delta_{\mathrm{even}}$ is the number of punctures whose asymptotic limit has an even δ-weighted index. Notice that (2.5) and (2.6) imply

$$0 \leq \mathrm{wind}_\pi(\tilde{u}) \leq c_N^\delta(\tilde{u}). \tag{2.7}$$

If $\delta_z = 0, \forall z \in \Gamma$, then $c_N^\delta(\tilde{u})$ is simply denoted $c_N(\tilde{u})$. If δ_z is not specified for some puncture z, then we tacitly assume that $\delta_z = 0$. If $m_0 = 0$, then $c_N^\delta(\tilde{u})$ is an integer, since $\mathrm{ind}^\delta(\tilde{u})$ and $\#\Gamma^\delta_{\mathrm{even}}$ have the same parity, see (2.4). The half-integer $c_N^\delta(\tilde{u})$ can be regarded as the algebraic number of zeros of a section $\sigma : \dot{\Sigma} \to N_{\tilde{u}}$ representing infinitesimal J-holomorphic variations of $\tilde{u}$ in the normal direction, keeping the same asymptotic limits at the punctures, the same boundary conditions, and respecting the weight constraints at the punctures. The zeros on the boundary $\partial\Sigma$ contribute half of their local degree. The section σ satisfies $D^N\bar{\partial}_J(\tilde{u}) \cdot \sigma = 0$, where $D^N\bar{\partial}_J\tilde{u}$ is the restriction and projection to the normal bundle $N_{\tilde{u}}$ of the linearized Fredholm operator $D\bar{\partial}_J(\tilde{u})$ between suitable weighted Sobolev spaces. The curve $\tilde{u}$ is said to be *regular* if $D\bar{\partial}_J(\tilde{u})$ is surjective. Any such a section of $N_{\tilde{u}}$ admits an asymptotic formula similar to the one in Theorem 2.1 so that at each puncture $z \in \Gamma$, σ has a leading eigenvalue and a leading eigenvector of the asymptotic operator A_{P_z}.

Theorem 2.5 (Wendl [71])**.** *Assume that $\tilde u$ is embedded. If $\operatorname{ind}^\delta(\tilde u) > c_N^\delta(\tilde u)$, then $\tilde u$ is regular. In particular, the space of J-holomorphic curves near $\tilde u$, with the same asymptotic limits, boundary conditions and δ-weight constraints, has the structure of a smooth manifold with dimension $\operatorname{ind}^\delta(\tilde u)$.*

Theorem 2.5 can be regarded as follows. Suppose by contradiction that an embedded curve $\tilde u$ with $\operatorname{ind}^\delta(\tilde u) > c_N^\delta(\tilde u)$ is not regular. Then the linearized Cauchy-Riemann operator in the normal direction has a non-trivial cokernel and thus the dimension of the kernel is $d \geq \operatorname{ind}^\delta(\tilde u) + 1 > c_N^\delta(\tilde u) + 1$. Considering d independent sections in the kernel, it is then possible to construct a non-trivial section with $d - 1 > c_N^\delta(\tilde u)$ zeros, a contradiction.

The curves satisfying the conditions of Theorem 2.5 are called automatically transverse. The following proposition is a direct consequence of Wendl's results in [69, 70, 71, 72].

Proposition 2.6. *The following assertions hold:*

(i) *If $\tilde u : \mathbb{D} \to \mathbb{R} \times \mathcal{M}$ is an embedded J-holomorphic disk so that $\mu(u^*\xi|_{\partial\mathbb{D}}, l_1) = 2$, then $\tilde u$ is regular, $\operatorname{ind}(\tilde u) = 1$, $\operatorname{wind}_\pi(\tilde u) = 0$ and $c_N(\tilde u) = 0$.*

(ii) *If $\tilde u : \mathbb{D}\backslash\{0\} \to \mathbb{R}\times\mathcal{M}$ is an embedded J-holomorphic punctured disk so that $\mu(u^*\xi|_{\partial\mathbb{D}}, l_1) = 2$, and 0 is a negative puncture whose asymptotic limit is an index-2 hyperbolic orbit, then $\operatorname{ind}(\tilde u) = 0$, $\operatorname{wind}_\pi(\tilde u) = 0$ and $c_N(\tilde u) = 0$.*

(iii) *If $\tilde u : \mathbb{C} \to \mathbb{R} \times \mathcal{M}$ is an embedded J-holomorphic plane whose asymptotic limit P_3 at ∞ has δ-weighted index $\mu^\delta(P_3) = 3$, then $\tilde u$ is regular, $\operatorname{ind}^\delta(\tilde u) = 2$, $\operatorname{wind}_\pi(\tilde u) = 0$ and $c_N^\delta(\tilde u) = 0$.*

(iv) *If $\tilde u : \mathbb{C}\setminus\{0\} \to \mathbb{R}\times\mathcal{M}$ is an embedded J-holomorphic cylinder whose asymptotic limit P_3 at the positive puncture ∞ has δ-weighted index $\mu^\delta(P_3) = 3$, and 0 is a negative puncture whose asymptotic limit is an index-2 hyperbolic orbit, then $\tilde u$ is regular, $\operatorname{ind}^\delta(\tilde u) = 1$, $\operatorname{wind}_\pi(\tilde u) = 0$ and $c_N^\delta(\tilde u) = 0$.*

Proof. We compute in each case

(i) $\operatorname{ind}(\tilde u) = 2 - 1 = 1, m_0 = 1$ and $\#\Gamma_{\text{even}} = 0$.

(ii) $\operatorname{ind}(\tilde u) = 2 - 2 - 0 = 0, m_0 = 1$, $\#\Gamma_{\text{even}} = 1$.

(iii) $\operatorname{ind}^\delta(\tilde u) = 3 - 1 = 2, m_0 = 0$ and $\#\Gamma^\delta_{\text{even}} = 0$.

(iv) $\operatorname{ind}^\delta(\tilde u) = 3 - 2 = 1$, $m_0 = 0$ and $\#\Gamma^\delta_{\text{even}} = 1$.

From (2.6), we obtain $c_N^\delta(\tilde u) = c_N(\tilde u) = 0$ in all cases. A direct application of inequality (2.7) gives $\operatorname{wind}_\pi(\tilde u) = 0$. In cases (i), (iii), and (iv), we see that $c_N^\delta(\tilde u) < \operatorname{ind}^\delta(\tilde u)$. By Theorem 2.5, $\tilde u$ is regular in those cases. □

Remark 2.7. *The curve $\tilde u : \mathbb{D}\setminus\{0\} \to \mathbb{R}\times\mathcal{M}$ in Proposition 2.6-(ii) is not automatically transverse. Theorem 1 in* [71] *implies that $\dim\ker D\bar\partial_J(\tilde u) \leq 1$ and it is not difficult to construct examples so that $\dim\ker D\bar\partial_J(\tilde u) = 1$. In that case, $\dim\operatorname{coker} D\bar\partial_J(\tilde u) = 1$ and thus $\tilde u$ is not regular.*

3. Proof of Theorem 1.3

Let $\mathcal{M}$, α, $\mathcal{P} = \{P_3, P_{2,1}, \ldots, P_{2,l}\} \subset \mathcal{P}(\alpha)$ and $J \in \mathcal{J}(\alpha)$ be as in Theorem 1.3. Recall that each $P_{2,i}$ lies in the sphere-like boundary component $\mathcal{S}_i \subset \partial\mathcal{M}$ and the hemispheres of $\mathcal{S}_i \setminus P_{2,i}$ are projections of a pair of nicely embedded J-holomorphic planes $\tilde u_{i,1} = (a_{i,1}, u_{i,1})$ and $\tilde u_{i,2} = (a_{i,2}, u_{i,2})$ asymptotic to $P_{2,i}$ and satisfying $u_{i,1}(\mathbb{C}) \cup u_{i,2}(\mathbb{C}) = \mathcal{S}_i \setminus P_{2,i}$.

The planes $\tilde u_{i,j}, j = 1, \ldots, l$, prevent families of J-holomorphic curves in $\mathcal{M}$ from escaping through $\partial\mathcal{M}$. They form a barrier so that the bubbling-off analysis is similar to that of a boundaryless contact three-manifold. For this reason, we call $\mathcal{S}_j$ a **spherical shield**.

Proposition 3.1. *Let $\tilde u = (a, u) : \dot\Sigma \to \mathbb{R}\times\mathcal{M}$ be a connected J-holomorphic curve with boundary conditions in $\mathcal{M}\setminus\partial\mathcal{M}$, and whose image is not contained in $\mathbb{R}\times\partial\mathcal{M}$. Then*

(i) *$\tilde u$ does not have a positive end at any orbit in $\partial\mathcal{M}$.*

(ii) *$u(\dot\Sigma) \subset \mathcal{M}\setminus\partial\mathcal{M}$.*

Proof. By contradiction, assume that $\tilde u = (a, u)$ has a positive end at $P_{2,i}^p$ for some $i \in \{1, \ldots, l\}$ and $p > 0$. Consider the asymptotic operator $A_{P_{2,i}^p}$ associated with the positive hyperbolic orbit

$P^p_{2,i}$. The leading eigenvalue of $\tilde{u}$ at the corresponding puncture must be the maximum negative eigenvalue $p\lambda_2 < 0$, where λ_2 is the maximum negative eigenvalue of $A_{P_{2,i}}$. In fact, otherwise the curve u winds around $P_{2,i}$ slower that the hemispheres of $\mathcal{S}_i \setminus P_{2,i}$, forcing the existence of points in $u(\dot{\Sigma})$ outside $\mathcal{M}$, a contradiction. The eigenspace associated with $p\lambda_2$ is one-dimensional and corresponds to the p-cover of the λ_2-eigenspace $\mathbb{R}e_2$ of $A_{P_{2,i}}$. In particular, e^p_2 is tangent to $\mathcal{S}_i$ along $P^p_{2,i}$. Consider the holomorphic plane $\tilde{v}$ that p-covers the hemisphere of $\mathcal{S}_i \setminus P_{2,i}$ and approaches $P^p_{2,i}$ through the same direction as $\tilde{u}$. After reparametrizing $\tilde{u}$ if necessary, both $\tilde{u}$ and $\tilde{v}$ have the same leading eigenvalues and eigenvectors. The maps u and v do not have the same image close to the corresponding punctures by assumption. Hence, Siefring's formula for the difference of J-holomorphic curves asymptotic to the same periodic orbit, see Theorem 2.2 in [66], implies that the difference between u and v in suitable coordinates is ruled by a leading eigenvalue $\lambda_d < 0$ strictly smaller than $p\lambda_2$. The winding number of λ_d is strictly less than the winding number of $p\lambda_2$. This implies that $\tilde{u}$ and $\tilde{v}$ wind around each other and, as before, $u(\dot{\Sigma})$ contains points outside $\mathcal{M}$, again a contradiction. This proves (i).

Now assume by contradiction that $u(\dot{\Sigma}) \cap \mathcal{S}_i \neq \emptyset$ for some i. Since every point in $\partial\mathcal{M}$ lies in the projection of a J-holomorphic curve, the assumptions on $\tilde{u}$ imply that the intersection of $\tilde{u}$ with the curves $\tilde{u}_{i,1}$, $\tilde{u}_{i,2}$ and the trivial cylinder $\mathbb{R} \times P_{2,i}$ are isolated. Moreover, $\int_{\dot{\Sigma}} u^* d\alpha > 0$ and a point $z \in \dot{\Sigma}$ so that $u(z) \in \mathcal{S}_i$ satisfies $z \in \dot{\Sigma} \setminus \partial\Sigma$. If $\tilde{u}(z) \in \mathbb{R} \times P_{2,i}$, then since $\tilde{u}$ satisfies a Cauchy-Riemann equation on the contact structure, a circle around z and close to z is mapped under u onto a closed curve that locally surrounds $P_{2,i}$, contradicting $u(\dot{\Sigma}) \subset \mathcal{M}$. By the similarity principle, this holds true even in the case $\pi \circ du(z) = 0$. Hence $\tilde{u}$ does not intersect $\mathbb{R} \times P_{2,i}$ and we may assume that $\tilde{u}$ intersects $\tilde{u}_{i,1}$. Again, these intersection points are isolated and $\tilde{u}$ has positive $d\lambda$-area. By positivity and stability of intersections, we may shift $\tilde{u}_{i,1}$ in the $\mathbb{R}$-direction, and $\tilde{u}$ intersects $r + \tilde{u}_{i,1}$ for all small values of $r > 0$. We keep increasing r and notice that the intersecting point in the domain of $\tilde{u}$ cannot escape through $\partial\Sigma$ by the standing assumption $u(\partial\Sigma) \subset \mathcal{M} \setminus \partial\mathcal{M}$. Also, it cannot escape through a negative puncture of $\tilde{u}$ since $r + a_{i,1}(\mathbb{C})$ is uniformly bounded from below for every $r > 0$. By the first statement of this proposition, the intersection point in the domain of $\tilde{u}$ cannot escape through a positive puncture. So there exists a compact subset $K \subset \dot{\Sigma} \setminus \partial\dot{\Sigma}$ that contains all points in the domain of $\tilde{u}$ corresponding to intersections between $\tilde{u}$ and $r + \tilde{u}_{i,1}$, for every $r > 0$. In the same way, the points in the domain of $r + \tilde{u}_{i,1}$ corresponding to intersections between $\tilde{u}$ and $r + \tilde{u}_{i,1}$ must be contained in a compact subset $K' \subset \mathbb{C}$ for every $r > 0$ since $a_{i,1} \to +\infty$ at ∞ uniformly. However, for $r > 0$ sufficiently large, we have $r + \min a_{i,1}(\mathbb{C}) > \max a(K)$, meaning that $\tilde{u}$ does not intersect $r + \tilde{u}_{i,1}$ for every $r > 0$ sufficiently large. This contradicts the assumption that $\tilde{u}$ intersects $\tilde{u}_{i,1}$, and (ii) is proved. □

3.1. **The Bishop family of J-holomorphic disks.** Let $P \subset \mathcal{M} \setminus \partial\mathcal{M}$ be a p-unknotted periodic orbit with self-linking number $-1/p$ and Conley-Zehnder index $\mu(P^p) \geq 3$. Assume that P bounds a p-disk $\mathcal{D}' \hookrightarrow \mathcal{M} \setminus \partial\mathcal{M}$. Consider the characteristic foliation $(T\mathcal{D}' \cap \xi)^\perp$, where $\perp$ is the symplectic complement with respect to any area form on $\mathcal{D}'$. We call $e'_0 \in \mathcal{D}'$ a singular point of the characteristic foliation if $T_{e'_0}\mathcal{D}' = \xi_{e'_0}$. We say a singular point $e'_0 \in \mathcal{D}' \setminus \partial\mathcal{D}'$ of the characteristic foliation is nicely elliptic if there exists a vector field $V : \mathcal{D}' \to (T\mathcal{D}' \cap \xi)^\perp$ so that $DV(e'_0)$ has positive eigenvalues. In particular, e'_0 is a source of V. Using Giroux's elimination procedure, it is always possible to C^0-perturb $\mathcal{D}'$ away from the boundary so that its characteristic foliation contains a unique singularity e_0 that is nicely elliptic. We may assume that the interior of $\mathcal{D}'$ contains no periodic orbit. We may also assume that $\mathcal{D}'$ is transverse to the Reeb vector field near P^p_3. Indeed, we may C^0-perturb $\mathcal{D}'$ in a tubular neighborhood of P_3 so that it approaches P^p_3 as an eigenvector of the asymptotic operator $A_{P^p_3}$ with winding number 1. Such an approach implies that $\mathcal{D}'$ is transverse to the Reeb vector field near P^p_3 and for each cross section at $x_0 \in P_3$, the p-disk $\mathcal{D}'$ is formed by p local branches approaching x_0, whose tangent at x_0 are not a positive multiple of each other.

Theorem 3.2 (Hofer [29])**.** *We may further C^0-perturb $\mathcal{D}'$ away from the boundary $\partial\mathcal{D}'$, keeping the previous properties, and C^∞-perturb J in a small neighborhood of e_0 so that the new disk $\mathcal{D}$*

and the new almost complex structure, still denoted by J, admits a family of J-holomorphic disks $\tilde{u}_\tau = (a_\tau, u_\tau) : (\mathbb{D}, i) \to \mathbb{R} \times \mathcal{M}, \tau \in (0, \epsilon), \epsilon > 0$ small, satisfying

$$
\begin{cases} \tilde{u}_\tau \text{ is an embedding}, a_\tau|_{\partial\mathbb{D}} \equiv 0, \\ u_\tau(\partial\mathbb{D}) \subset \mathcal{D} \setminus (\partial\mathcal{D} \cup \{e_0\}), \\ \operatorname{wind}(u_\tau|_{\partial\mathbb{D}}, e_0) = +1. \end{cases} \tag{3.1}
$$

Finally, $\lim_{\tau\to 0^+} u_\tau(z) = e_0$ uniformly in $z \in \mathbb{D}$.

One of the key properties of the J-holomorphic disk $\tilde{u} = \tilde{u}_\tau$ given in Theorem 3.2 is that $u|_{\partial\mathbb{D}}$ is transverse to the characteristic foliation. Indeed, take polar coordinates (θ, r) on $\mathbb{D}$. Since $\Delta a = |\pi u_s|^2 \geq 0$, the strong maximum principle implies that $(\partial_r a)|_{\partial\mathbb{D}} > 0$. Hence $\alpha(\partial_\theta u|_{\partial\mathbb{D}}) > 0$ and thus $u|_{\partial\mathbb{D}}$ is positively transverse to the leaves of $(T\mathcal{D} \cap \xi)^\perp$. Since $\tilde{u}$ is an embedding and $a|_{\partial\mathbb{D}} \equiv 0$, we conclude from $\operatorname{wind}(u_\tau|_{\partial\mathbb{D}}, e_0) = +1$ that $u|_{\partial\mathbb{D}}$ is an embedding that intersects each leaf of the characteristic foliation precisely once.

Another important property of the family u_τ is that if $\tau < \tau'$, then $u_\tau(\partial\mathbb{D})$ is contained in the interior of the disk in $\mathcal{D}$ bounded by $u_{\tau'}(\partial\mathbb{D})$. Indeed, the three-dimensional group of biholomorphisms of $(\mathbb{D}, i)$ acts freely on the space of J-holomorphic disks satisfying (3.1). For each τ, the linearization $D\bar{\partial}_{\tilde{J}}(\tilde{u}_\tau)$ is a surjective Fredholm operator with Fredholm index 4. Since the group of automorphisms of the disk has dimension 3, the unparametrized space $\mathcal{M}_0$ of J-holomorphic disks satisfying the boundary conditions in (3.1) is one-dimensional, see Proposition 2.6-(i).

Let $V \subset T\mathcal{D}$ be a vector field generating $(T\mathcal{D} \cap \xi)^\perp$, and let $N_{\tilde{u}_\tau} \subset \tilde{u}_\tau^* T(\mathbb{R} \times \mathcal{M})$ be a normal complex line bundle satisfying $N_{\tilde{u}_\tau} \oplus T_{\tilde{u}_\tau} = \tilde{u}_\tau^* T(\mathbb{R} \times \mathcal{M})$, so that $N_{\tilde{u}_\tau}$ coincides with the contact structure and thus contains $0 \oplus \mathbb{R}V$ along $\partial\mathbb{D}$. Infinitesimally, the curves near $\tilde{u}_\tau \in \mathcal{M}_0$ are modeled by non-trivial sections $\sigma : \mathbb{D} \to N_{\tilde{u}_\tau}$ satisfying $\sigma|_{\partial\mathbb{D}} \subset \mathbb{R}V$. Indeed, Proposition 2.6-(i) gives $c_N^\delta(\tilde{u}_\tau) = 0$ and thus σ never vanishes. Hence, the nearby disks in the Bishop family do not intersect each other, see [29], and $u_\tau(\partial\mathbb{D})$ is strictly monotone towards $P_3^p = \partial\mathcal{D}$ as τ increases. Moreover, $\pi \circ du_\tau$ never vanishes since $0 \leq \operatorname{wind}_\pi(\tilde{u}_\tau) \leq c_N(\tilde{u}_\tau) = 0$ and thus u_τ is transverse to the Reeb vector field.

3.2. **Compactness properties.** Before studying the compactness properties of the Bishop family, we assume that the contact form α and the almost complex structure J satisfy the following C^∞-generic conditions:

- H1. α is nondegenerate up to action $\mathcal{S}(\mathcal{D}, \alpha) = \int_{\mathcal{D}} |d\alpha|$. Notice that the energy of any $\tilde{u}_\tau$ satisfying (3.1) is bounded by $\mathcal{S}(\mathcal{D}, \alpha)$.
- H2. Let $\tilde{u} = (a, u) : \mathbb{D} \setminus \Gamma \to \mathbb{R} \times (\mathcal{M} \setminus \partial\mathcal{M})$, $\emptyset \neq \Gamma \subset \mathbb{D} \setminus \partial\mathbb{D}$, be a somewhere injective J-holomorphic punctured disk satisfying the analogous conditions in (3.1), and so that at the negative punctures in Γ, $\tilde{u}$ is asymptotic to orbits with index ≥ 2. Then $\#\Gamma = 1$ and $\tilde{u}$ is asymptotic to an index-2 orbit at its negative puncture. Moreover, $\tilde{u}$ is Fredholm regular.
- H3. Let $\tilde{u} = (a, u) : \mathbb{C} \setminus \Gamma \to \mathbb{R} \times (\mathcal{M} \setminus \partial\mathcal{M})$, $\Gamma \neq \emptyset$, be a somewhere injective J-holomorphic curve with non-vanishing $d\alpha$-area, asymptotic to P_3^p at ∞, whose leading eigenvalue has winding number $+1$, and to orbits in $\partial\mathcal{M}$ at the negative punctures in Γ. Then $\#\Gamma = 1$ and $\tilde{u}$ is asymptotic to some index-2 orbit at its negative puncture.

Condition H1 is achieved under C^∞-small perturbations of α, see Proposition 6.1 in [36]. Fixing $\mathcal{D}$, condition H2 is achieved for C^∞-generic $J \in \mathcal{J}(\alpha)$, see [35, Theorem 1.14] and [19]. Indeed, let $\tilde{u} : \mathbb{D} \setminus \Gamma \to \mathbb{R} \times (\mathcal{M} \setminus \partial\mathcal{M})$ be as in H2. If $\#\Gamma \geq 2$ or there exists a negative puncture where $\tilde{u}$ is asymptotic to an orbit with index ≥ 3, then $\operatorname{ind}(\tilde{u}) < 2 - 2\#\Gamma - \chi(\mathbb{D}) + \#\Gamma = 1 - \#\Gamma \leq 0$. Since $\tilde{u}$ is somewhere injective, such a curve does not exist for a generic J. We conclude that $\tilde{u}$ has precisely one puncture whose asymptotic limit has index 2. Moreover, $\tilde{u}$ is Fredholm regular for a C^∞-generic J. Condition H3 also holds for a C^∞-generic J. In fact, such a curve is necessarily somewhere injective and as in H2, for a generic J, its weighted index is at least 1. Hence, as before, it has at most 1 negative puncture and the asymptotic limit at this negative puncture has index 2.

From now on we assume that α and J satisfy conditions H1-H3. The set of such J's for a fixed α is denoted by $\mathcal{J}_{\rm reg}(\alpha) \subset \mathcal{J}(\alpha)$. Notice that the assumptions of Theorem 1.3 still hold if the perturbation of α is sufficiently small in the C^∞-topology. Indeed, we restrict to perturbations that keep P_3 as a periodic orbit, and the condition $\mu(P_3^p) \geq 3$ still holds. Moreover, if $P' \subset \mathcal{M} \setminus (\partial\mathcal{M} \cup P_3)$ is contractible and satisfies $\rho(P') = 1$ and $\mathrm{link}(P', P_3^p) = 0$, then its action must be $> \mathcal{S}(\mathcal{D}, \alpha)$. Otherwise, we find a sequence of such nondegenerate perturbations $\alpha_n \to \alpha$ in C^∞ and periodic orbits P_n of α_n, satisfying $\rho(P_n) = 1$, $\mathrm{link}(P_n, P_3^p) = 0$ and $\mathcal{A}(P_n) \leq \mathcal{S}(\mathcal{D}, \alpha_n)$, and converging to a periodic orbit $P \subset \mathcal{M} \setminus (\partial\mathcal{M} \cup P_3)$ of α which is contractible and satisfies $\rho(P) = 1$, $\mathrm{link}(P, P_3^p) = 0$ and $\mathcal{A}(P) \leq \mathcal{S}(\mathcal{D}, \alpha)$, a contradiction. Notice that P cannot coincide with a cover of P_3 since P is contractible and its rotation number is 1.

Since the disks in the Bishop family are automatically transverse, the family $\tilde{u}_\tau$ persists under C^∞-small perturbations of J and thus we may assume they exist for some $J \in \mathcal{J}_{\rm reg}(\alpha)$.

We choose three distinct leaves l_1, l_i and l_{-1} of the characteristic foliation $(T\mathcal{D} \cap \xi)^\perp$ issuing from e_0 to $\partial\mathcal{D}$, in the counterclockwise direction, and reparametrize the disks $\tilde{u}_\tau$ in the Bishop family so that

$$u_\tau(z) \in l_z, \quad \forall z \in \{1, i, -1\}, \quad \forall \tau \in (0,1). \tag{3.2}$$

This parametrization goes back to [29] and controls the compactness properties of the Bishop family. Take the maximal family $\tilde{u}_\tau = (a_\tau, u_\tau), \tau \in (0,1)$ of such J-holomorphic disks. By Proposition 3.1, $u_\tau(\mathbb{D}) \subset \mathcal{M} \setminus \partial\mathcal{M}$ for every $\tau \in (0,1)$.

Proposition 3.3. *Under the conditions H1-H3, the maximal family $\tilde{u}_\tau, \tau \in (0,1)$, SFT-converges as $\tau \to 1$ to one of the following buildings $\mathcal{B}$:*

(i) *$\mathcal{B} = (\tilde{v}_1, \tilde{v}_2)$ has two levels. The top level $\tilde{v}_1$ is a nicely embedded punctured disk $\tilde{v}_1 = (a_1, v_1) : \mathbb{D} \setminus \{0\} \to \mathbb{R} \times \mathcal{M}$ satisfying the boundary conditions in* (3.1)*, with a negative puncture at* 0 *asymptotic to some $P_{2,i} \subset \partial\mathcal{M}$. The lowest level v_2 is an embedded plane asymptotic to $P_{2,i}$ that projects to one of the hemispheres of $\mathcal{S}_i \setminus P_{2,i}$.*

(ii) *$\mathcal{B} = (\tilde{v}_1, \tilde{v}_2)$ has two levels. The top level is a punctured disk $\tilde{v}_1 : \mathbb{D} \setminus \{0\} \to \mathbb{R} \times \mathcal{M}$ whose image is $(-\infty, 0] \times P_3$. The lowest level is a nicely embedded plane $\tilde{v}_2$ asymptotic to P_3^p, and projecting to $\mathcal{M} \setminus \partial\mathcal{M}$.*

(iii) *$\mathcal{B} = (\tilde{v}_1, \tilde{v}_2, \tilde{v}_3)$ has three levels. The top level v_1 is a punctured disk $\tilde{v}_1 : \mathbb{D} \setminus \{0\}$ whose image is $(-\infty, 0] \times P_3$. The second level is a nicely embedded cylinder $\tilde{v}_2 : \mathbb{R} \times \mathbb{R}/\mathbb{Z} \to \mathbb{R} \times \mathcal{M}$ with a positive end at P_3^p and a negative end at some $P_{2,i} \subset \partial\mathcal{M}$. The lowest level consists of an embedded plane $\tilde{v}_3 : \mathbb{C} \to \mathbb{R} \times \mathcal{M}$ asymptotic to $P_{2,i}$ that projects to one of the hemispheres of $\mathcal{S}_i \setminus P_{2,i}$.*

Moreover, the building in (i) occurs if and only if $\inf_\tau$ *dist*$(u_\tau(\partial\mathbb{D}), \partial\mathcal{D}) > 0$.

Proof. The parametrization (3.2) prevents bubbling-off points at the boundary $\partial\mathbb{D}$. In fact, it follows from [31, Theorem 2.1], see also [29], that there exists $\epsilon > 0$ small such that $\sup\{|\nabla \tilde{u}_\tau(z)| : 1 - \epsilon < |z| \leq 1, \tau \in (0,1)\} < +\infty$. Hence $\tilde{u}_\tau$ only admits bubbling-off points in the interior of $\mathbb{D}$. Consider a sequence $\tau_n \to 1^-$ as $n \to \infty$ and denote $\tilde{u}_n = (a_n, u_n) := \tilde{u}_{\tau_n}$. Since every bubbling-off point takes away at least $\gamma/2 > 0$ of the $d\alpha$-area ($\gamma > 0$ is the shortest period of a periodic orbit) and $\int_{\mathbb{D}} \tilde{u}_n^* d\alpha$ is uniformly bounded by $\mathcal{S}(\mathcal{D}, \alpha)$, we find a finite set $\Gamma \subset \mathbb{D} \setminus \partial\mathbb{D}$ so that, up to a subsequence, $\tilde{u}_n \to \tilde{v}_1 = (b_1, v_1) : \mathbb{D} \setminus \Gamma \to \mathbb{R} \times \mathcal{M}$ in $C^\infty_{\rm loc}$ as $n \to +\infty$. Every puncture of $\tilde{v}_1$ is necessarily negative.

Assume first the $\inf_\tau \mathrm{dist}(u_\tau(\partial\mathbb{D}), \partial\mathcal{D}) > 0$. Since $\mathcal{D} \setminus \partial\mathcal{D}$ contains no periodic orbits, we conclude that $\tilde{v}_1$ satisfies the boundary conditions in (3.1) and its $d\alpha$-area is positive. If $\Gamma = \emptyset$, then $\tilde{v}_1$ is an embedded disk since it is the limit of embeddings and $\tilde{v}_1$ is embedded near the boundary. Indeed, recall that $\partial_r b_1 > 0$ and $v_1|_{\partial\mathbb{D}}$ transversely intersects $(T\mathcal{D} \cap \xi)^\perp$. Since $\mathrm{wind}(v_1|_{\partial\mathbb{D}}, e_0) = \mathrm{wind}(u_n|_{\partial\mathbb{D}}, e_0) = +1$ for every n, it follows that $v_1|_{\partial\mathbb{D}}$ and $\tilde{v}_1$ are embedded near the boundary. By the automatic transversality of the disks in the Bishop family, see Proposition 2.6-(i), we conclude that $\tilde{v}_1$ lies in the interior of a unique local family of such disks containing $\tilde{u}_n$, n large, contradicting the maximality of the family $\tilde{u}_\tau, \tau \in (0,1)$. Hence $\Gamma \neq \emptyset$. Since the energy of $\tilde{u}_n$

is uniformly bounded by $\mathcal{S}(\mathcal{D},\alpha)$ and α is nondegenerate up to action $\mathcal{S}(\mathcal{D},\alpha)$, see condition H1, we can invoke the SFT compactness theorem to obtain a building $\mathcal{B}$ which is the SFT-limit of $\tilde{u}_n$, up to a subsequence, so that the top level of $\mathcal{B}$ is $\tilde{v}_1$. Notice that $\tilde{v}_1$ is somewhere injective since it is embedded near the boundary. The curves below $\tilde{v}_1$ have precisely one positive puncture and an arbitrary number of negative punctures. Also, every asymptotic limit of a curve in $\mathcal{B}$ is contractible and thus its index is well-defined. Moreover, for every curve in $\mathcal{B}$, the action of every asymptotic limit is uniformly bounded by $\mathcal{S}(\mathcal{D},\alpha)$. If necessary, we may see the disks in the Bishop family lifted to the symplectization $\mathbb{R}\times S^3$ whose almost complex structure is the one lifted from $\mathbb{R}\times L(p,q)$ under $\pi_{p,q}:S^3\to L(p,q)$. The same holds for the limiting curves in $\mathcal{B}$ after taking a subsequence. In particular, the asymptotic limits have a well-defined Conley-Zehnder index computed in a symplectic global frame of $\xi\to S^3$.

First, we prove that all asymptotic limits of $\tilde{v}_1$ have index ≥ 2. This argument is found in [41]. Suppose by contradiction that $\tilde{v}_1$ has an asymptotic limit γ with index ≤ 1 at a negative puncture in Γ. Then $\mathcal{B}$ contains a curve $\tilde{v}_2:\mathbb{C}\setminus\Gamma_2\to\mathbb{R}\times\mathcal{M}$ below $\tilde{v}_1$, and positively asymptotic to γ at ∞. All punctures in Γ_2 are negative, and their asymptotic limits are contractible. We claim that $\tilde{v}_2$ has an asymptotic limit at a negative puncture whose index is ≤ 1. If the $d\alpha$-area of $\tilde{v}_2$ vanishes, then $\Gamma_2\neq\emptyset$ and $\tilde{v}_2$ is a k-branched cover of the trivial cylinder over a contractible Reeb orbit $\hat{\gamma}$. We may assume that $\#\Gamma_2\geq 2$, otherwise $\tilde{v}_2$ is a trivial cylinder and the claim trivially holds. Since the index of $\gamma=\hat{\gamma}^k$ is ≤ 1, the index of $\hat{\gamma}$ is necessarily ≤ 1. Thus, the asymptotic limit at a negative puncture of $\tilde{v}_2$ has the form $\hat{\gamma}^m$ for some $m<k$. Hence, $\hat{\gamma}^m$ has index ≤ 1. Now, assume that the $d\alpha$-area of $\tilde{v}_2$ is positive. Let $\text{wind}_\infty(z)$ be the winding number of the leading eigenvalue of $\tilde{v}_2$ at $z\in\Gamma\cup\{\infty\}$. Since the index of γ is ≤ 1, we have $\text{wind}_\infty(\infty)\leq 0$. This implies that $0\leq\text{wind}_\pi(\tilde{v}_2)=\text{wind}_\infty(\infty)-\sum_{z\in\Gamma_2}\text{wind}_\infty(z)-(1-\#\Gamma_2)\leq -1-\sum_{z\in\Gamma_2}(\text{wind}_\infty(z)-1)$. It follows that there exists $z\in\Gamma_2$ such that $\text{wind}_\infty(z)\leq 0$. This implies that γ_z has index ≤ 0. We conclude in both cases that $\tilde{v}_2$ has an asymptotic limit at a negative puncture with index ≤ 1. Similarly, we find a curve $\tilde{v}_3\in\mathcal{B}$ below $\tilde{v}_2$ with a negative puncture whose asymptotic limit has index ≤ 1. Continuing this procedure, we eventually find a J-holomorphic plane in a leaf of $\mathcal{B}$, which is asymptotic to a Reeb orbit with index ≤ 1. However, such a plane cannot exist, and we conclude that every asymptotic limit of $\tilde{v}_1$ has index ≥ 2.

Next, we claim that $\#\Gamma=1$ and $\tilde{v}_1$ is asymptotic to an index-2 orbit $P_2=P_{2,i}\subset\partial M$ for some i. Since $\tilde{v}_1$ is somewhere injective, it follows from condition H2, that $\tilde{v}_1$ has precisely one negative puncture whose asymptotic limit P_2 has index 2. Hence $\tilde{v}_1$ is a punctured J-holomorphic disk, and we may assume, after parametrization, that $\Gamma=\{0\}$. Since P_2 is contractible and has rotation number 1, it is geometrically distinct from P_3. This implies that the $d\alpha$-area of $\tilde{v}_1$ is positive. Also, P_2 is unlinked with P_3^p since the image of any circle $S\subset\mathbb{D}$ under u_n is unlinked with P_3^p. Hence, the same holds for $\tilde{v}_1$. Moreover, since the action of P_2 is $\leq\mathcal{S}(\mathcal{D},\alpha)$, we conclude from our linking hypothesis that P_2 coincides with one of the Lyapunov orbits, say $P_2=P_{2,i_0}$ for some $i_0\in\{1,\dots,l\}$. In particular, $\tilde{v}_1$ is an embedded punctured disk asymptotic to P_{2,i_0} at its negative puncture.

The curve $\tilde{v}_2=(v_2,b_2):\mathbb{C}\setminus\Gamma_2\to\mathbb{R}\times\mathcal{M}$ below $\tilde{v}_1$ is asymptotic to P_{2,i_0} at its positive puncture ∞. By Proposition 3.1-(i), $\tilde{v}_2$ must coincide with one of the curves projecting to a hemisphere of $\mathcal{S}_{i_0}\setminus P_{2,i_0}$. We may assume that $\tilde{v}_2=\tilde{u}_{i_0,1}$ and the building $\mathcal{B}$ has no other levels besides the ones formed by $\tilde{v}_1$ and $\tilde{v}_2$. Thus $\mathcal{B}$ is as in (i).

Now assume that $\inf_\tau\text{dist}(u_\tau(\partial\mathbb{D}),\partial\mathcal{D})=0$, and pick $\tilde{u}_n:=\tilde{u}_{\tau_n}$ with $\tau_n\to 1$ as $n\to\infty$. As before, no bubbling-off point occurs at the boundary $\partial\mathbb{D}$. Moreover, the first level of $\mathcal{B}$ is a punctured disk $\tilde{v}_1=(b_1,v_1):\mathbb{D}\setminus\Gamma_1\to\mathbb{R}\times M$, $\Gamma_1\subset\mathbb{D}\setminus\partial\mathbb{D}$, so that v_1 intersects P_3. If $\tilde{v}_1$ has positive $d\alpha$-area, then an important fact proved in [31, Theorem 4.4] using a degree argument implies that $\tilde{u}_n$ intersects $\mathbb{R}\times P_3$ for n sufficiently large, a contradiction. Hence the $d\alpha$-area of $\tilde{v}_1$ vanishes and $\tilde{v}_1$ is a trivial half-cylinder over P_3^p whose image is $(-\infty,0]\times P_3$. This follows from the fact that $\mathcal{D}$ is a p-disk for P_3 and thus $u_n(\partial\mathbb{D})$ converges to P_3^p as $n\to\infty$. The curve $\tilde{v}_2=(b_2,v_2):\mathbb{C}\setminus\Gamma_2\to\mathbb{R}\times M$ below $\tilde{v}_1$ is asymptotic to P_3^p at its positive puncture ∞. We may assume that $\tilde{v}_2$ is not a trivial cylinder over P_3^p. Also, it cannot be a branched cover of a trivial cylinder over $P_3^{p_0}$, p_0 divides p, since P_3^j is non-contractible for every $0<j=mp_0<p, m\in\mathbb{N}$,

and all asymptotic limits of $\tilde{v}_2$ are contractible. Hence, the $d\alpha$-area of $\tilde{v}_2$ is positive, and the leading eigenvalue of $\tilde{v}_2$ at ∞ has winding number 1 with respect to a global symplectic frame of $\xi \to S^3$. This follows from the fact that the lift of P_3 to S^3 is a $\mathbb{Z}_p$-symmetric trivial knot $\hat{P}_3 \equiv P_3^p$ and the lift of the image of any circle $S \subset \mathbb{D}$ under $\tilde{u}_n$ is unlinked with $\hat{P}_3$. In particular, the leading eigenvector, when projected to P_3^p is simple in the sense that it does not cover an eigenvector along a smaller iterate of P_3. This implies that $\tilde{v}_2$ is somewhere injective. Arguing as before, the index of every asymptotic limit of $\tilde{v}_2$ at a negative puncture is ≥ 2. The inequality $0 \leq \text{wind}_\pi(\tilde{v}_2) = \text{wind}_\infty(\infty) - \sum_{z\in\Gamma_2} \text{wind}_\infty(z) - (1 - \#\Gamma_2) = -\sum_{z\in\Gamma_2}(\text{wind}_\infty(z) - 1)$ implies that P_z has index 2 for every $z \in \Gamma_2$. Indeed, notice that $\text{wind}_\infty(z) \geq 2$ if P_z has index ≥ 3. Since P_z is unlinked with P_3 and the only orbits with action $\leq \mathcal{S}(\mathcal{D}, \alpha)$ that are unlinked with P_3 are the orbits in ∂M, we conclude that $P_z \subset \partial\mathcal{M}, \forall z \in \Gamma_2$. Our generic choice of J implies that $\#\Gamma_2 \in \{0, 1\}$, see condition H3.

If $\Gamma_2 = \emptyset$, then $\tilde{v}_2$ is an embedded plane asymptotic to P_3^p, and not intersecting $\mathbb{R} \times P_3$. Its projection v_2 is an embedding, see Theorem 5.20 in [67]. By Proposition 3.1, $v_2(\mathbb{C}) \subset \mathcal{M} \setminus \partial\mathcal{M}$. The building $\mathcal{B} = (\tilde{v}_1, \tilde{v}_2)$ is as in (ii). If $\#\Gamma_2 = 1$, let P_{2,i_0} be the asymptotic limit at the negative puncture in Γ_2. The level $\tilde{v}_3$ below $\tilde{v}_2$ must then be a plane asymptotic to P_{2,i_0} projecting to a hemisphere of $\mathcal{S}_{i_0} \setminus P_{2,i_0}$, see Proposition 3.1 As limits of embedded curves, both $\tilde{v}_2$ and $\tilde{v}_3$ are embedded and do not intersect any of its asymptotic limits. Therefore, their projections are also embedded, see Theorem 5.20 in [67]. This gives the building in (iii). □

Proposition 3.4. *There exists a sequence of J-holomorphic disks $\tilde{u}_n = (a_n, u_n) : \mathbb{D} \to \mathbb{R} \times \mathcal{M}$ satisfying the conditions in* (3.1)*, and so that* $\inf_n$ *dist*$(u_n(\partial\mathbb{D}), \partial\mathcal{D}) = 0$*. Moreover, $\tilde{u}_n$ admits an SFT-limit as in Proposition 3.3-(ii) or (iii).*

Proof. Let us assume that the maximal family $\tilde{u}_\tau, \tau \in (0, 1)$, issuing from the nicely elliptic singularity of $(T\mathcal{D} \cap \xi)^\perp$ for $\tau = 0$, satisfies $\inf_\tau \text{dist}(u_\tau(\partial\mathbb{D}), \partial\mathcal{D}) > 0$. Then it breaks into a building $\mathcal{B} = (\tilde{u}_1, \tilde{v}_1)$ as in Proposition 3.3-(i). Moreover, $\tilde{u}_1 : \mathbb{D}\setminus\{0\} \to \mathbb{R}\times\mathcal{M}$ is an embedded J-holomorphic punctured disk satisfying the boundary conditions in (3.1), it is asymptotic at $0 \in \mathbb{D}$ to $P_{2,i} \subset \mathcal{S}_i \subset \partial\mathcal{M}$ for some i, and $\tilde{v}_1 : \mathbb{C} \to \mathbb{R}\times\mathcal{M}$ is one of the planes $\tilde{u}_{i,1}$ or $\tilde{u}_{i,2}$. Let us assume without loss of generality that $\tilde{v}_1 = \tilde{u}_{i,1}$. Our generic choice of J implies that $\tilde{u}_1$ is a regular curve. Since $\tilde{u}_{i,1}$ is also an embedded regular curve, we can glue the embedded curves $\tilde{u}_1$ and $\tilde{u}_{i,1}$, see [46, 47], to obtain a new family of embedded J-holomorphic disks, denoted $\tilde{u}_\tau, \tau \in (1, 1+\epsilon), \epsilon > 0$ small.

Lemma 3.5. *Every disk $\tilde{u}_\tau = (a_\tau, u_\tau), \tau \in (1, 1+\epsilon)$, satisfies* (3.1)*. Moreover, $u_\tau(\mathbb{D}) \subset \mathcal{M}\setminus\partial\mathcal{M}$, and $u_\tau(\partial\mathbb{D})$ lies in the exterior of $u_1(\partial\mathbb{D}) \subset \mathcal{D}$.*

Proof. Since $\tilde{u}_1$ and $\tilde{u}_{i,2}$ are embedded and their generalized intersection number vanishes, the glued curves $\tilde{u}_\tau$ are also embedded. The conditions $a_\tau|_{\partial\mathbb{D}} \equiv 0$, $u_\tau(\partial\mathbb{D}) \subset \mathcal{D} \setminus (\partial\mathcal{D} \cup \{e_0\})$ and $\text{wind}(u_\tau|_{\partial\mathbb{D}}, e_0) = +1$ follow from similar properties of $\tilde{u}_1$. We also know that $\text{wind}_\pi(\tilde{u}_\tau) = 0$ and thus u_τ is transverse to the flow.

To prove that $u_\tau(\mathbb{D}) \subset \mathcal{M} \setminus \partial\mathcal{M}, \forall\tau - 1 > 0$ sufficiently small, we assume by contradiction that $u_\tau(\mathbb{D}) \cap \partial\mathcal{M} \neq \emptyset$ for some $\tau - 1 > 0$ arbitrarily small. Since $\partial\mathcal{M}$ is the union of projections of J-holomorphic curves in $\mathbb{R}\times\partial\mathcal{M}$, $\tilde{u}_\tau$ admits only isolated intersections with a curve $\tilde{v}$ projecting to $\mathcal{S}_j \subset \partial\mathcal{M}$ for some j. Since u_τ is transverse to the flow, we may assume that $\tilde{v} = (b, v) : \mathbb{C} \to \mathbb{R}\times\mathcal{M}$ projects to one of the hemispheres of $\mathcal{S}_j \setminus P_{2,j}$. Notice that $u_\tau(\partial\mathbb{D})$ and $P_{2,j}$ are unlinked and $v(\mathbb{C}) \cap u_\tau(\partial\mathbb{D}) = \emptyset$. Consider the shifted curves $\tilde{v}_a := a + \tilde{v}, a \in \mathbb{R}$. Notice that $u_\tau(\partial\mathbb{D})$ does not intersect $v_a(\mathbb{C}) = v(\mathbb{C}) \subset \partial\mathcal{M}$. Since $a_\tau|_\mathbb{D} \leq 0$ and $b(z) \to +\infty$ as $z \to \infty$, the points in $\mathbb{C}$ and $\mathbb{D}$ corresponding to intersections between $\tilde{v}_a, a \geq 0$, and $\tilde{u}_\tau$ stay uniformly bounded in $\mathbb{C}$ and inside some disk $\mathbb{D}_r = \{z \in \mathbb{C} : |z| \leq r\} \subset \mathbb{D}$, $0 < r < 1$, respectively. However, for $a > 0$ sufficiently large, such intersections cannot exist since $a_\tau|_\mathbb{D} \leq 0$ and $a + \min_{z\in\mathbb{C}} b(z) \to +\infty$ as $a \to +\infty$. From the stability and positivity of intersections between pseudo-holomorphic curves, intersections between $\tilde{u}_\tau$ and $\tilde{v}_a$ cannot exist for any $a \geq 0$, a contradiction.

To prove that $u_\tau(\partial\mathbb{D})$ lies in the exterior of $u_1(\partial\mathbb{D}) \subset \mathcal{D}\setminus\partial\mathcal{D}$, we need the following result from [3] concerning uniqueness of pseudo-holomorphic discs.

Proposition 3.6 (Abbas-Hofer [3, Theorem 7.6.3]). *Let $\tilde{v}, \tilde{u}_\tau : \mathbb{D} \to \mathbb{R} \times M, \tau \in (-1,1)$, satisfy (3.1). Assume that $\tilde{u}_\tau(\mathbb{D}) \cap \tilde{v}(\mathbb{D}) = \emptyset$ for every $\tau \in (0,\epsilon)$, and that $\tilde{u}_0(\mathbb{D}) \cap \tilde{v}(\mathbb{D}) \neq \emptyset$. Then $\tilde{v} = \tilde{u}_0$, up to a bi-holomorphism of $\mathbb{D}$.*

We know that $\tilde{u}_\tau(\mathbb{D}) \cap \tilde{u}_1(\mathbb{D} \setminus \{0\}) = \emptyset$ for every $\tau < 1$ sufficiently close to 1. This follows from the fact that the sections of the normal bundle representing infinitesimal variations of the disks in the Bishop family never vanish. The same holds for the family $\tilde{u}_\tau$ for $\tau > 1$ sufficiently close to 1. It follows from this fact and $\tilde{u}_\tau(\mathbb{D}) \subset \mathcal{M} \setminus \partial\mathcal{M}$ that if we fix $\tau_0 < 1$ arbitrarily close to 1, then for every $\tau > 1$ sufficiently close to 1, we have $\tilde{u}_{\tau_0}(\mathbb{D}) \cap \tilde{u}_\tau(\mathbb{D}) = \emptyset$.

Let us assume by contradiction that $u_\tau(\partial\mathbb{D}), \tau > 1$, lies in the same component of $\mathcal{D} \setminus u_1(\partial\mathbb{D})$ as $u_\tau(\partial\mathbb{D}), \tau < 1$, i.e., in the disk-like region bounded by $u_1(\partial\mathbb{D}) \subset \mathcal{D}$. We may fix $\tau_0 < 1$ sufficiently close to 1 and choose $\tau_0^* > 1$ sufficiently close to 1 so that $\tilde{u}_{\tau_0}(\mathbb{D}) \cap \tilde{u}_{\tau_0^*}(\mathbb{D}) \neq \emptyset$. We may assume that $\tilde{u}_{\tau_0}(\mathbb{D}) \cap (\tilde{u}_1(\mathbb{D} \setminus \{0\}) \cup (\mathbb{R} \times \partial\mathcal{M})) = \emptyset$ and $\tilde{u}_{\tau_0^*}(\mathbb{D}) \cap (\tilde{u}_1(\mathbb{D} \setminus \{0\}) \cup (\mathbb{R} \times \partial\mathcal{M})) = \emptyset$ as explained before. From the strict monotonicity of the circles $u_\tau(\partial\mathbb{D}) \subset \mathcal{D}$, we know that there exists τ_1^* satisfying $1 < \tau_1^* < \tau_0^*$ such that for every $1 < \tau < \tau_1^*$ we have

- $\tilde{u}_\tau(\mathbb{D}) \cap \tilde{u}_{\tau_0}(\mathbb{D}) = \emptyset$. In particular, $u_\tau(\partial\mathbb{D}) \cap u_{\tau_0}(\partial\mathbb{D}) = \emptyset$.
- $u_\tau(\partial\mathbb{D})$ lies in the annulus in $\mathcal{D}$ bounded by $u_{\tau_0}(\partial\mathbb{D})$ and $u_1(\partial\mathbb{D})$.

Hence we find $\tau_2^* > 1$ satisfying $1 < \tau_1^* < \tau_2^* < \tau_0^*$ such that $\tilde{u}_{\tau_2^*}(\mathbb{D}) \cap \tilde{u}_{\tau_0}(\mathbb{D}) \neq \emptyset$ and $\tilde{u}_\tau(\mathbb{D}) \cap \tilde{u}_{\tau_0}(\mathbb{D}) = \emptyset$ for every $1 < \tau < \tau_2^*$. By Proposition 3.6, this implies that $\tilde{u}_{\tau_2^*}(\mathbb{D}) = \tilde{u}_{\tau_0}(\mathbb{D})$. This is a contradiction since for $\tau_0 < 1 < \tau_0^*$ sufficiently close to 1, $u_{\tau_0}(\mathbb{D})$ is arbitrarily close to $u_1(\mathbb{D} \setminus \{0\}) \cup P_{2,j} \cup v_1(\mathbb{C})$ and $u_{\tau_2^*}(\mathbb{D})$ is arbitrarily close to $u_1(\mathbb{D} \setminus \{0\}) \cup (\mathcal{S}_j \setminus v_1(\mathbb{C}))$. We conclude that $u_\tau(\partial\mathbb{D})$ lies in the exterior of $u_1(\partial\mathbb{D}) \subset \mathcal{D}$. □

We continue the proof of Proposition 3.4. Consider the maximal family $\tilde{u}_\tau = (a_\tau, u_\tau), \tau \in (1,2)$, satisfying (3.1). Notice that $\tilde{u}_\tau$ is embedded for every τ, $\tilde{u}_\tau(\mathbb{D}) \cap \tilde{u}_{\tau'}(\mathbb{D}) = \emptyset, \forall \tau \neq \tau'$, and if $\tau < \tau'$ then $u_\tau(\partial\mathbb{D})$ is contained in the interior of $u_{\tau'}(\partial\mathbb{D}) \subset \mathcal{D}$. If $\inf_{\tau \in (1,2)} \operatorname{dist}(u_\tau(\partial\mathbb{D}), \partial\mathcal{D}) = 0$, then the proof of Proposition 3.4 is finished. Otherwise, take a sequence $\tilde{w}_n = (c_n, w_n) := \tilde{u}_{\tau_n}, \tau_n \to 2$, and apply Proposition 3.3-(i) to obtain a new embedded half-cylinder $\tilde{u}_2 = (a_2, u_2) : \mathbb{D} \setminus \{0\} \to \mathbb{R} \times \mathcal{M}$ and a plane $\tilde{v}_2 = (b_2, v_2) : \mathbb{C} \to \mathbb{R} \times \mathcal{M}$ so that $u_2(\mathbb{D}) \setminus \{0\} \subset \mathcal{M} \setminus \partial\mathcal{M}$ and $v_2(\mathbb{C}) \subset \mathcal{S}_j \subset \partial\mathcal{M}$ for some j. Gluing $\tilde{u}_2$ with the plane projecting to the hemisphere $\mathcal{S}_j \setminus (P_{2,j} \cup v_2(\mathbb{C}))$ as before, we obtain a new family of embedded disks $\tilde{u}_\tau = (a_\tau, u_\tau), \tau \in (2, 2+\epsilon)$, satisfying (3.1). Using Proposition 3.6, we conclude that $u_\tau(\partial\mathbb{D})$ lies in the exterior of $u_2(\partial\mathbb{D}) \subset \mathcal{D}$ and the monotonicity of the boundary $u_\tau|_{\partial\mathbb{D}}$ still holds. Repeating this procedure either after finitely many steps we find a maximal family $\tilde{u}_\tau, \tau \in (k, k+1)$ satisfying (3.1) and so that $\inf_{\tau \in (k,k+1)} \operatorname{dist}(u_\tau(\partial\mathbb{D}), \partial\mathcal{D}) = 0$, or, after a gluing procedure as above, we obtain countably many families $\tilde{u}_\tau, \tau \in (k+1, k+2), k \in \mathbb{N}$. In the first case, the proof of Proposition 3.4 is finished. In the second case, since every half-cylinders $\tilde{w} : \mathbb{D} \setminus \{0\} \to \mathbb{R} \times \mathcal{M}$ satisfying (3.1) and asymptotic to some $P_{2,i}$ is regular, $\tilde{w}$ is isolated in the space of such half-cylinders with the same asymptotic limit at the negative puncture. Hence, such half-cylinders cannot accumulate away from $\partial\mathcal{D}$ and thus Proposition 3.3 tells us that we necessarily have $\inf_{k \in \mathbb{N}} \operatorname{dist}(u_k(\partial\mathbb{D}), \partial\mathcal{D}) = 0$. Hence, the desired sequence of J-holomorphic disks is also obtained in this case. □

By Proposition 3.4, either we find a nicely embedded J-holomorphic plane $\tilde{u} = (a,u) : \mathbb{C} \to \mathbb{R} \times \mathcal{M}$ asymptotic to P_3^p, or a nicely embedded J-holomorphic cylinder $\tilde{v} = (b,v) : \mathbb{C} \setminus \{0\} \to \mathbb{R} \times \mathcal{M}$ with a positive puncture at ∞ asymptotic to P_3^p, whose leading eigenvalue has winding number 1, and a negative puncture at 0 asymptotic to some $P_{2,j} \subset \mathcal{S}_j$. We must have $u(\mathbb{C}) \subset \mathcal{M} \setminus (\partial\mathcal{M} \cup P_3)$ and $v(\mathbb{C} \setminus \{0\}) \subset \mathcal{M} \setminus (\partial\mathcal{M} \cup P_3)$ in each case, see Proposition 3.1. In the second case, we can glue $\tilde{v}$ with a J-holomorphic curve projecting to one of the hemispheres in $\mathcal{S}_j \setminus P_{2,j}$ to obtain a holomorphic plane $\tilde{u} = (a,u) : \mathbb{C} \to \mathbb{R} \times \mathcal{M}$ asymptotic to P_3^p such that $u(\mathbb{C}) \subset \mathcal{M} \setminus (\partial\mathcal{M} \cup P_3)$ as in the first case. Indeed, in the gluing procedure, we may consider only sections of the normal bundle that converge to P_3^p with sufficiently large exponential decay, so that the glued curve $\tilde{u}$ approaches P_3^p at ∞ with a leading eigenvalue that has winding number 1 with respect to the global trivialization of the contact structure. In particular, intersections with P_3^p are not created from infinity and $\operatorname{wind}_\pi(\tilde{u}) = \operatorname{wind}_\infty(\infty) - 1 = 0$, implying that u is transverse to the flow.

The fact that u is an embedding follows from standard intersection theory of pseudo-holomorphic curves, see [42, 67]. We have proved the following proposition.

Proposition 3.7. *There exists an embedded J-holomorphic plane $\tilde{u} = (a, u) : \mathbb{C} \to \mathbb{R} \times \mathcal{M}$ asymptotic to P_3^p, so that its projection $u : \mathbb{C} \to \mathcal{M} \setminus (\partial\mathcal{M} \cup P_3^p)$ is embedded and transverse to the flow. The winding of the leading eigenvalue of $\tilde{u}$ at ∞ is $+1$.*

In the case $p = 1$, these are the fast planes considered in [38, 39]. The case $p > 1$ was treated in [40, 42]. Let $\mathcal{M}_{P_3^p}$ denote the set of J-holomorphic planes $\tilde{u} = (a, u) : \mathbb{C} \to \mathbb{R} \times \mathcal{M}$ asymptotic to P_3^p and satisfying

$$\begin{cases} \tilde{u} \text{ and } u \text{ are embeddings,} \\ u(\mathbb{C}) \subset \mathcal{M} \setminus (\partial\mathcal{M} \cup P_3^p), \\ \mathrm{wind}_\pi(\tilde{u}) = 0. \end{cases}$$

The δ-weighted index of any $\tilde{u} \in \mathcal{M}_{P_3^p}$ is $\mathrm{ind}_\delta(\tilde{u}) = \mu_\delta(\tilde{u}) - (\chi(\mathbb{C}P^1) - 1) = 3 - 1 = 2$. With this weight, $\tilde{u}$ is automatically transverse, and $\mathcal{M}_{P_3^p}$ is a two-dimensional manifold containing all the $\mathbb{R}$-translations of $\tilde{u}$. One can cut out a transverse one-parameter family of $\tilde{u}_\tau = (a_\tau, u_\tau) \in \mathcal{M}_{P_3^p}, \tau \in (-\epsilon, \epsilon)$, with $\tilde{u}_0 = \tilde{u}$, so that

$$u_\tau(\mathbb{C}) \cap u_{\tau'}(\mathbb{C}) = \emptyset, \quad \forall \tau \neq \tau'. \tag{3.3}$$

Consider the maximal family $\tilde{u}_\tau \in \mathcal{M}_{P_3^p}, \tau \in (-1, 1)$, satisfying (3.3), where maximality means that $\cup_\tau u_\tau(\mathbb{C}) \subset \mathcal{M} \setminus \partial\mathcal{M}$ has the largest volume.

Proposition 3.8. *Under the conditions above, the maximal family $\tilde{u}_\tau, \tau \in (0, 1)$, converges in the SFT sense as $\tau \to 1$ (or as $\tau \to -1$) to a two-level building $\mathcal{B} = (\tilde{u}_1, \tilde{v}_1)$, so that $\tilde{u}_1$ is a nicely embedded cylinder $\tilde{u}_1 = (a_1, u_1) : \mathbb{C} \setminus \{0\} \to \mathbb{R} \times \mathcal{M}$ with a positive puncture at ∞ asymptotic to $P_3^p = (x_3, pT_3)$, whose leading eigenvalue has winding number $+1$, and a negative puncture at 0 asymptotic to some $P_{2,i} \subset \partial\mathcal{M}$. The lowest level $\tilde{v}_1$ is a plane asymptotic to $P_{2,i}$ and projecting to one of the hemispheres of $\mathcal{S}_i \setminus P_{2,i}$.*

Proof. Take a sequence $\tilde{u}_n = (a_n, u_n) := \tilde{u}_{\tau_n}$, with $\tau_n \to 1$. Since the energy of $\tilde{u}_n$ is constant equal to pT_3 and α is nondegenerate up to action $\mathcal{S}(\mathcal{D}, \alpha) \geq pT_3$, see condition H1, there exists a building $\mathcal{B}$ which is the SFT-limit of $\tilde{u}_n$, up to a subsequence. The first level of $\mathcal{B}$ is a curve $\tilde{v}_1 : \mathbb{C} \setminus \Gamma \to \mathbb{R} \times \mathcal{M}$, with a positive puncture at ∞, where it is asymptotic to P_3^p. The asymptotic limits of $\tilde{v}_1$ at the negative punctures in Γ are contractible. The $d\alpha$-area of $\tilde{v}_1$ is positive since smaller iterates P_3^j, $j < p$ are non-contractible and $\tilde{v}_1$ is not a trivial cylinder. Hence, the leading eigenvalue of $A_{P_3^p}$ has winding number $+1$. In particular, $\tilde{v}_1$ is somewhere injective. Every puncture $z \in \Gamma$ is negative and the index of its asymptotic limit P_z is 2 and thus unlinked with P_3, see the proof of Proposition 3.3.

We claim that $\#\Gamma = 1$ and $\tilde{v}_1$ is asymptotic to some $P_{2,i} \subset \partial\mathcal{M}$. Indeed, if $\Gamma = \emptyset$, then $\tilde{v}_1 \in \mathcal{M}_{P_3^p}$, contradicting the maximality of the family $\tilde{u}_\tau$. Hence $\#\Gamma \geq 1$. From the generic choice of J, see condition H3, we have $\#\Gamma = 1$, and we can assume that $\Gamma = \{0\}$. Since the action of P_2 is $< pT_3$, the assumptions on α imply that $\tilde{v}_1$ is asymptotic to some $P_{2,i} \subset \partial\mathcal{M}$. By Proposition 3.1, the curve $\tilde{v}_2$ below $\tilde{v}_1$ is an embedded J-holomorphic plane asymptotic to $P_{2,i}$ and projecting to a hemisphere of $\mathcal{S}_i \setminus P_{2,i}$. The embedding properties of $\tilde{u}_n$, see (3.3), imply that $\mathcal{B}$ is the unique SFT-limit of the family $\tilde{u}_\tau$. □

We can assume that the family $\tilde{u}_\tau, \tau \in (0, 1)$, is parametrized so that $u_\tau(\mathbb{C})$ moves in the direction of the flow as τ increases. By Proposition 3.8, the family $\tilde{u}_\tau$ breaks onto a nicely embedded cylinder $\tilde{u}_1$ asymptotic to P_3^p at the positive puncture and asymptotic to $P_{2,j} \subset \mathcal{S}_j$ at the negative puncture, plus a plane $\tilde{v}_2$ that projects to a hemisphere of $\mathcal{S}_j \setminus P_{2,j}$. Let $\tilde{v}_1'$ be the plane asymptotic to $P_{2,j}$ projecting to the other hemisphere of $\mathcal{S}_j$. Using the appropriate weights at P_3^p, we can glue $\tilde{u}_1$ with $\tilde{v}_1'$, see [46, 47], to obtain a family of planes $\tilde{u}_\tau \in \mathcal{M}_{P_3^p}, \tau \in (1, 1 + \epsilon)$, asymptotic to P_3^p at ∞, whose leading eigenvalue has winding number $+1$. Their projections u_τ to $\mathcal{M} \setminus \partial\mathcal{M}$ do not intersect each other and also do not intersect the corresponding maps of the family $\tilde{u}_\tau$, for $\tau \in (0, 1)$. Considering the maximal family $\tilde{u}_\tau, \tau \in (1, 2)$, we have two possibilities.

If $l = 1$, i.e. $\partial\mathcal{M}$ has only one boundary component, then for τ close to 2, $\tilde{u}_\tau$ coincides with some $\tilde{u}_{\tau'}$ for some $\tau' < 1$ close to 1. Indeed, this follows from the uniqueness of cylinders $\tilde{u}_1$ from P_3^p to $P_{2,1}$ and the uniqueness of planes obtained from gluing $\tilde{u}_1$ with the hemispheres of $\mathcal{S}_1$, see Theorem 2.3. If $l > 1$, then, again by the uniqueness of cylinders and planes, the family $\tilde{u}_\tau$ as $\tau \to 2$ cannot break onto $\tilde{u}_1$ and a plane projecting to $\mathcal{S}_j$. Indeed, otherwise, the projection of u_τ, $\tau < 1$ sufficiently close to 1 is homotopic to $u_{\tau'}$, $\tau' > 1$ sufficiently close to 1. This is not possible for topological reasons. Hence the family $\tilde{u}_\tau, \tau \in (1,2)$ breaks into a new cylinder $\tilde{u}_2 \neq \tilde{u}_1$ connecting P_3^p to $P_{2,i} \subset \mathcal{S}_i$, for some $i \neq j$, and a plane $\tilde{v}_2$ asymptotic to $P_{2,i}$ projecting to $\mathcal{S}_i$. We can continue this procedure of gluing $\tilde{u}_2$ with the opposite plane $\tilde{v}_2'$ projecting to the other hemisphere of $\mathcal{S}_i \setminus P_{2,i}$, to extend the family of planes asymptotic to P_3^p. The number of breaks is necessarily equal to l, and we end up finding l families of embedded planes $\{\tilde{u}_\tau\}_{\tau\in(i,i+1)}$, $i = 1,\dots,l$, all asymptotic to P_3^p, their projections are also embedded and mutually disjoint. We also obtain l embedded cylinders $\tilde{v}_i, i = 1,\dots,l$, connecting P_3^p at the positive puncture to a distinct $P_{2,j} \subset \mathcal{S}_j$ at the negative puncture. Their projections are embedded, and the union of such curves, with the ones projecting to $\partial\mathcal{M}$, determines a finite energy foliation of $\mathbb{R}\times\mathcal{M}$ whose projection to $\mathcal{M}$ is a weakly convex foliation. Notice that the families of planes asymptotic to P_3^p and the cylinders connecting P_3^p to each $P_{2,i}$ for an open and close subset of $\mathcal{M}\setminus\partial\mathcal{M}$ thus foliate the whole $\mathcal{M}\setminus\partial\mathcal{M}$.

3.3. **Passing to the degenerate case.** Let $\mathcal{M},\alpha,\mathcal{P}$ and J be as in Theorem 1.3. In particular, the hemispheres of $\mathcal{S}_i \setminus P_{2,i}$ are the projections of two holomorphic planes $\tilde{u}_{i,1},\tilde{u}_{i,2}$, for every $i = 1,\dots,l$. After a C^∞-small perturbation of J supported in $\mathcal{M}\setminus\partial\mathcal{M}$, we may assume that condition H3 is satisfied. We denote by $\mathcal{J}_{\rm reg}(\alpha)$ the space of J's satisfying condition H3.

Now let α_n be a sequence of nondegenerate contact forms on (M,ξ_0), coinciding with α on a small neighborhood of $\partial\mathcal{M}$ and converging in C^∞ to α so that P_3 is a Reeb orbit of α_n for every n. Let $J \in \mathcal{J}_{\rm reg}(\alpha)$ be such that the hemispheres in $\mathcal{S}_i \setminus P_{2,i}$ are the projections of J-holomorphic planes $\tilde{u}_{i,1},\tilde{u}_{i,2}$. Let $J_n \in \mathcal{J}(\alpha_n)$ be a sequence converging to J in C^∞ so that conditions H1-H3 are satisfied for (α_n, J_n), where in condition H2 the p-disk for P_3 is independent of n. We may assume that J_n coincides with J on a neighborhood of $\partial\mathcal{M}$. In particular, the hemispheres of $\mathcal{S}_i \setminus P_{2,i}$ are projections of J_n-holomorphic planes for every n and i.

Proposition 3.9. *Under the conditions above, the following holds: for every n sufficiently large, (α_n, J_n) satisfies the condition $\mathcal{P}' = \emptyset$ in Theorem 1.3, i.e., α_n does not admit a contractible periodic orbit $P' \subset \mathcal{M}\setminus(\partial\mathcal{M}\cup P_3)$ unlinked with P_3, with rotation number 1 and action $\leq \mathcal{S}(\mathcal{D},\alpha_n)$. In particular, (α_n, J_n) admits a finite energy foliation projecting to a weakly convex foliation whose binding orbits are $P_3 \in \mathcal{M}\setminus\partial\mathcal{M}$ and $P_{2,1},\dots,P_{2,l} \subset \partial\mathcal{M}$.*

Proof. Fix a p-disk $\mathcal{D}$ for P_3 so that the characteristic foliation $(\xi_0 \cap T\mathcal{D})^\perp$ has a unique singularity, which is nicely elliptic. Since $\alpha_n \to \alpha$, we have $\mathcal{S}(\mathcal{D},\alpha_n) \to \mathcal{S}(\mathcal{D},\alpha)$ as $n \to \infty$. We claim that for every n sufficiently large, α_n does not admit a contractible index-2 Reeb orbit $P_n \subset \mathcal{M}\setminus\partial\mathcal{M}$ with action $\leq \mathcal{S}(\alpha_n,\mathcal{D})$ and unlinked with P_3. By contradiction, assume that such an orbit exists for n arbitrarily large. Up to a subsequence $P_n \to Q$, where $Q \subset \mathcal{M}$ is a contractible periodic orbit of α with rotation number 1 and action $\leq \mathcal{S}(\mathcal{D},\alpha)$. Since each $P_{2,i} \subset \partial\mathcal{M}$ is a hyperbolic orbit of α, Q cannot coincide with some $P_{2,i}$, and thus $Q \subset \mathcal{M}\setminus\partial\mathcal{M}$. Also, Q is not a cover of P_3 since the rotation number of any contractible cover of P_3 is > 1. Hence $Q \subset \mathcal{M}\setminus(\partial\mathcal{M}\cup P_3)$ has rotation number 1 and is unlinked with P_3, contradicting the hypotheses on α. □

Consider a sequence $\alpha_n \to \alpha$ and $J_n \to J$ as in Proposition 3.9, and let $\mathcal{F}_n$ be the weakly convex foliation adapted to (α_n, J_n) given in Theorem 1.3 so that its binding orbits are $P_3, P_{2,1},\dots,P_{2,l}$. Then the J_n-holomorphic planes projecting to $\partial\mathcal{M}$ do not depend on n and hence we only need to consider the compactness properties of the family of planes asymptotic to P_3^p and the cylinders connecting P_3^p at the positive end to $P_{2,i}$ at the negative end. Before doing that, we need to show that α does not admit any periodic orbit which is unlinked with P_3, regardless of its index and knot-type.

Proposition 3.10. *The contact form α admits no contractible periodic orbit $P' \subset \mathcal{M}\setminus(\partial\mathcal{M}\cup P_3)$ which is unlinked with P_3.*

Proof. Suppose by contradiction the existence of a contractible periodic orbit P' of α as in the statement. Then we can construct a sequence of nondegenerate contact forms $\alpha_n \to \alpha$ satisfying all conditions above with the additional property that P' is a periodic orbit of α_n for every n. Choose $J_n \in \mathcal{J}_{\rm reg}(\alpha_n)$ converging to J as $n \to \infty$. Proposition 3.9 tells us that (α_n, J_n) admits a weakly convex foliation whose binding orbits are $P_3, P_{2,1}, \ldots, P_{2,l}$ for n sufficiently large. This implies, in particular, that every periodic orbit of α_n in $\mathcal{M} \setminus (\partial\mathcal{M} \cup P_3)$ is linked with P_3, a contradiction. $\square$

Now, let us prove that rigid cylinders of $\mathcal{F}_n$ converge to a rigid cylinder for (α, J) with the same asymptotic limits. Denote by $\tilde{v}_n = (b_n, v_n) : \mathbb{R} \times \mathbb{R}/\mathbb{Z} \to \mathbb{R} \times M$ the nicely embedded J_n-holomorphic cylinder asymptotic to P_3^p at its positive puncture $+\infty$ and to $P_{2,i}$ at its negative puncture $-\infty$. We know that $v_n(\mathbb{R} \times \mathbb{R}/\mathbb{Z}) \subset \mathcal{M} \setminus \partial\mathcal{M}$ for every n and the leading eigenvalues at $+\infty$ and $-\infty$ have winding number $+1$. Fix a small compact tubular neighborhood $U \subset M$ of P_3 satisfying the following conditions for α

(i) U has no periodic orbit which is contractible in U;
(ii) There exists no periodic orbit $P' \subset U$ which is geometrically distinct from P_3, homotopic to P_3^p in U and unlinked with P_3.

Parametrize $\tilde{v}_n$ so that

$$\begin{cases} v_n(\{0\} \times \mathbb{R}/\mathbb{Z}) \cap \partial U \neq \emptyset, \\ v_n(s,t) \in U \setminus \partial U, \forall s > 0, \\ b_n(1,0) = 0. \end{cases} \tag{3.4}$$

Proposition 3.11. *Up to a subsequence, $\tilde{v}_n$ converges in C^∞_{loc} to a J-holomorphic cylinder $\tilde{w}_i = (b_i, w_i) : \mathbb{R}\times\mathbb{R}/\mathbb{Z} \to \mathbb{R}\times M$ asymptotic to P_3^p at its positive puncture $+\infty$, whose leading eigenvalue has winding number $+1$, and to $P_{2,i}$ at its negative puncture $-\infty$.*

Proof. From the normalization (3.4), we can extract a subsequence of $\tilde{v}_n$ in $C^\infty_{\rm loc}$ converging to a J-holomorphic punctured cylinder $\tilde{w}_i = (b_i, w_i) : (\mathbb{R} \times \mathbb{R}/\mathbb{Z}) \setminus \Gamma \to \mathbb{R} \times \mathcal{M}$ which is necessarily asymptotic to P_3^p at its positive puncture $+\infty$, see [17, Lemma 3.12]. The hypotheses on U imply that no bubbling occurs on $\{s > 0\}$. We must have $w_i((0, +\infty) \times \mathbb{R}/\mathbb{Z}) \subset U$ and thus $\Gamma \subset (-\infty, 0] \times \mathbb{R}/\mathbb{Z}$. Also, $-\infty$ is necessarily a negative puncture of $\tilde{w}_i$. We shall prove that $\Gamma = \emptyset$ and $\tilde{w}_i$ is asymptotic to $P_{2,i}$ at $-\infty$.

If the $d\alpha$-area of $\tilde{w}_i$ vanishes, then $\Gamma = \emptyset$ and $\tilde{w}_i$ is asymptotic to P_3^p at $-\infty$. This follows from the fact that all asymptotic limits of $\tilde{w}_i$ are contractible and every iterate P_3^j, $1 \leq j < p$, is non-contractible. However, in this case, we have $w_i(0,0) \in \partial U$, a contradiction. Hence, the $d\alpha$-area of $\tilde{w}_i$ is positive.

Let us prove that $\Gamma = \emptyset$ and $\tilde{w}_i$ is asymptotic to $P_{2,i}$ at $-\infty$. We know that $+\infty$ is a nondegenerate puncture. The leading eigenvalue of $A_{P_3^p}$ has winding number $+1$. Indeed, since $\mu(P_3^p) \geq 3$, the leading eigenvalue of $\tilde{v}_n$ at $+\infty$ has winding number 1 and stays away from 0 as $\alpha_n \to \alpha$. Hence, as proved in [36], see also [15, Chapter 8], the limiting curve $\tilde{w}_i$ admits a negative leading eigenvalue whose eigenvector has winding number $+1$, which describes the asymptotic behavior of $\tilde{w}_i$ at $+\infty$ as in Theorem 2.1.

Any asymptotic limit at a negative puncture in $\Gamma \cup \{-\infty\}$ must be contractible in $\mathcal{M}$, its action is $< pT_3$, and it is unlinked with P_3. By Proposition 3.10, the only orbits with these properties are the covers of orbits in $\partial\mathcal{M}$. Hence, every puncture of $\tilde{w}_i$ is nondegenerate. If $\#\Gamma \geq 1$, then the weighted index of $\tilde{w}_i$ is $\operatorname{ind}^\delta(\tilde{w}_i) \leq 3 - 2(\#\Gamma + 1) - (2 - \#\Gamma - 2) = 1 - \#\Gamma \leq 0$, contradicting the choice of $J \in \mathcal{J}_{\rm reg}(\alpha)$, see condition H3. Hence $\Gamma = \emptyset$. To prove that $\tilde{w}_i$ is asymptotic to $P_{2,i}$ at $-\infty$, we invoke the SFT-compactness theorem. In fact, this is possible since any asymptotic limit of any J-holomorphic curve arising from rescaling $\tilde{v}_n$ near $-\infty$, which is not P_3^p, has action $< pT_3$, is contractible in $\mathcal{M}$, and is unlinked with P_3. Hence, it necessarily lies in $\partial\mathcal{M}$, and consists of a hyperbolic orbit. The asymptotic limit of $\tilde{w}_i$ at $-\infty$ necessarily has index 2. Otherwise, the index of $\tilde{w}_i$ is < 0, again a contradiction with the choice of J. If the asymptotic limit of $\tilde{w}_i$ is $P_{2,j}$, $j \neq i$, then there exists a nontrivial building $\mathcal{B}$ below $\tilde{w}_i$ formed by J-holomorphic curves asymptotic to covers of orbits in $\partial\mathcal{M}$, so that $\mathcal{B}$ has a positive puncture at $P_{2,j}, j \neq i$, and a negative puncture at $P_{2,i}$. In particular, the first level of $\mathcal{B}$ consists of a J-holomorphic punctured sphere $\tilde{w} : \mathbb{C} \setminus \Gamma' \to \mathbb{R} \times \mathcal{M}$, which is not a trivial cylinder over $P_{2,j}$, and thus has positive $d\alpha$-area. It is asymptotic to $P_{2,j}$ at its unique positive puncture and to covers of orbits in $\partial \mathcal{M}$ at the negative punctures. However, by Proposition 3.1, such a curve $\tilde{w}$ does not exists. Hence the asymptotic limit of $\tilde{w}_i$ at $-\infty$ is $P_{2,i}$. □

Let us consider a sequence of J_n-holomorphic planes $\tilde{u}_n = (a_n, u_n) : \mathbb{C} \to \mathbb{R} \times \mathcal{M}$. As in the cylinder case, we may choose a small neighborhood $U \subset \mathcal{M}$ of P_3 as before, and parametrize $\tilde{u}_n$ so that

$$\begin{cases} u_n(z_n^*), u_n(1) \in \partial U, \text{ where } \mathrm{Re}(z_n^*) \leq 0. \\ u_n(\mathbb{C} \setminus \mathbb{D}) \subset U \setminus \partial U, \\ a_n(2) = 0. \end{cases} \tag{3.5}$$

This parametrization is obtained by considering the smallest disk in $\mathbb{C}$ that contains $u_n^{-1}(\mathcal{M} \setminus U)$ and suitably re-scaling it to $\mathbb{D}$.

Proposition 3.12. *Up to a subsequence, $\tilde{u}_n$ converges to one of the following buildings $\mathcal{B}$:*

(i) *$\mathcal{B}$ has only one level, which is a J-holomorphic plane $\tilde{u} = (a, u) : \mathbb{C} \to \mathbb{R} \times \mathcal{M}$ asymptotic to P_3^p. Moreover, $\tilde{u}$ is an embedding, u is an embedding transverse to the flow, $u(\mathbb{C}) \cap P_3 = \emptyset$, and the leading eigenvalue of $\tilde{u}$ at ∞ has winding number $+1$.*

(ii) *$\mathcal{B} = (\tilde{v}, \tilde{u})$ has two levels, the first level $\tilde{v} = (b, v) : \mathbb{R} \times \mathbb{R}/\mathbb{Z} \to \mathbb{R} \times \mathcal{M}$ is an embedded J-holomorphic cylinder asymptotic to P_3^p at its positive puncture and to some $P_{2,i} \subset \mathcal{S}_i$ at its negative puncture. Also, v is an embedding transverse to the flow and $v(\mathbb{R} \times \mathbb{R}/\mathbb{Z}) \subset \mathcal{M} \setminus (\partial \mathcal{M} \cup P_3)$. The curve $\tilde{v}_2$ is a J-holomorphic plane projecting to a hemisphere of $\mathcal{S}_i \setminus P_{2,i}$.*

Proof. From the normalization (3.5), we know that up to a subsequence $\tilde{u}_n$ converges in C^∞_{loc} to a J-holomorphic curve $\tilde{v} = (b, v) : \mathbb{C} \setminus \Gamma \to \mathbb{R} \times \mathcal{M}$, where $\Gamma \subset \mathbb{D}$ is a finite set of negative punctures, and $\tilde{v}$ is asymptotic to P_3^p at ∞. Moreover, $+\infty$ is a nondegenerate puncture and the leading eigenvalue describing the approach of $\tilde{v}$ to P_3^p has winding number $+1$ as in the proof of Proposition 3.11. Also, the $d\alpha$-area of $\tilde{v}$ is positive. Indeed, if it vanishes, then $\Gamma \neq \emptyset$ and $\tilde{v}$ is a trivial cylinder over P_3^p. This contradicts $v(1) \in \partial U$. Hence, $\int_{\mathbb{C} \setminus \Gamma} v^* d\alpha > 0$. Any asymptotic limit of a negative puncture in Γ is contractible, has action $< pT_3$, and is unlinked with P_3. By Proposition 3.10, such a Reeb orbit coincides with a cover of an orbit in $\partial \mathcal{M}$. As in the proof of Proposition 3.11, if $\#\Gamma > 1$, the weighted Fredholm index of $\tilde{v}$ is ≤ 0, a contradiction with the choice of J, see condition H3. If $\Gamma = \emptyset$, then $\tilde{v}$ is a plane as in (i). It is immediate that $\tilde{v}$ is an embedding, v is an embedding transverse to the flow, and $v(\mathbb{C}) \subset \mathcal{M} \setminus (\partial \mathcal{M} \cup P_3)$ since the same properties hold for the curves $\tilde{u}_n$ in the sequence. If $\#\Gamma = 1$, then by the SFT-compactness theorem, $\tilde{u}_n$ converges up to a subsequence to a building $\mathcal{B}$ with at least two levels. The first level contains only $\tilde{v} = (b, v)$, as above. Also, $\tilde{v}$ is asymptotic to P_3^p at ∞ and to some $P_{2,i}$ at its unique puncture in Γ. The level below $\tilde{v}$, consists of a J-holomorphic curve $\tilde{v}_2 = (b_2, v_2) : \mathbb{C} \setminus \Gamma' \to \mathbb{R} \times \mathcal{M}$, asymptotic to $P_{2,i}$ at ∞ and to other orbits in $\partial \mathcal{M}$ at its negative punctures in Γ'. Notice that $\tilde{v}_2$ is not a trivial cylinder and this fact implies that $\int_{\mathbb{C} \setminus \Gamma'} v_2^* d\alpha > 0$. If $\Gamma' \neq \emptyset$, then $\tilde{v}_2$ does not coincide with a hemisphere of $\mathcal{S}_i \setminus P_{2,i}$, contradiction Proposition 3.1. Hence, $\Gamma' = \emptyset$ and the building $\mathcal{B}$ is as in (ii). □

Propositions 3.11 and 3.12 imply that given $q \in \mathcal{M} \setminus (\partial \mathcal{M} \cup P_3)$, there exists either a J-holomorphic cylinder $\tilde{v} = (b, v)$ connecting P_3^p to some $P_{2,i} \subset \partial M$ whose projection to $\mathcal{M}$ passes through q, or a J-holomorphic plane $\tilde{u} = (a, u)$ asymptotic to P_3^p with the same property. Both $\tilde{v}$ and $\tilde{u}$ admit a leading negative eigenvalue at ∞ whose winding number is $+1$. The weighted Fredholm theory developed for embedded curves implies that such a plane asymptotic to P_3^p lies in a smooth family of planes whose projections to $\mathcal{M}$ are either identical or disjoint, see [38, 39]. The cylinders $\tilde{v}$ are rigid, in the sense that they are isolated in the space of such cylinders. The orbit $P_3, P_{2,1}, \ldots, P_{2,l}$ together with the families of planes asymptotic to P_3^p, the rigid cylinders, and the curves projecting to $\partial \mathcal{M}$ cover the whole manifold $\mathcal{M}$ forming a weakly convex foliation with binding orbits $P_3, P_{2,1}, \dots, P_{2,l}$. In fact, by uniqueness of cylinders connecting P_3^p to $P_{2,i}$, see Theorem 2.3-(ii), there exist precisely l rigid cylinders $\tilde{v}_i = (b_i, v_i) : \mathbb{R}\times\mathbb{R}/\mathbb{Z} \to \mathbb{R}\times\mathcal{M}, i = 1, \dots, l,$ asymptotic to P_3^p at the positive puncture $+\infty$ and to $P_{2,i}$ at the negative puncture $-\infty$. The family of planes $\tilde{u}_{i,\tau}, i \in \{1, \dots, l\}, \tau \in (0, 1)$ must break as $\tau \to 1^-$ onto a building consisting of a cylinder connecting P_3^p to $P_{2,i}$ and $\tilde{u}_{i,1}$, which projects to $\mathcal{S}_i$. Gluing $\tilde{v}_i$ with $\tilde{u}_{i,2}$, we obtain a family of nicely embedded planes $\tilde{u}_{i+1,\tau}, \tau \in (0, 1)$, which coincide with the planes in $\tilde{u}_{i,\tau}$ if and only if $l = 1$. If $l > 1$, then this new family will break onto a building with a cylinder $\tilde{v}_{i+1}$ connecting P_3^p to $P_{2,i+1}$ and a J-holomorphic plane $\tilde{u}_{i+1,1}$, after suitably labeling the rigid cylinders and rigid planes. We obtain a new family of planes by gluing $\tilde{v}_{i+1}$ with $\tilde{u}_{i+1,2}$. This stops after obtaining the l families of planes, and the weakly convex foliation adapted to (α, J) is finally constructed. This completes the proof of Theorem 1.3.

3.4. **Proof of Corollary 1.4.** The proof of Corollary 1.4 follows from the ideas in [37]. Let $i \in \{1, \dots, l\}$ be such that $P_{2,i}$ has the largest action $T_i > 0$ among the actions of $P_{2,1}, \dots, P_{2,l}$. Consider the local branch W_i^u of the unstable manifold of $P_{2,i}$ that lies in the interior of $\mathcal{M}$. Consider the family of planes $D_\tau, \tau \in (0, 1)$, in $\mathcal{F}$ so that as $\tau \to 0^+$, D_τ breaks into a hemisphere U_i of $\mathcal{S}_i$ and a cylinder connecting P_3^p to $P_{2,i}$. Assume that the Reeb vector field along U_i points inward $\mathcal{M}$ and thus τ increases in the direction of the flow. In particular, W_i^u intersects D_τ for $\tau > 0$ sufficiently small, and this intersection is a smooth circle C bounding a compact disk $B \subset D_\tau$, whose symplectic area is $T_i > 0$. The forward flow pushes B to a disk $B_1 \subset D_{\tau'}$, where $\tau' < 1$ is arbitrarily close to 1. Observe that for $\tau' < 1$ sufficiently close to 1, $D_{\tau'}$ intersects the local branch W_{i+1}^s of the stable manifold of $P_{2,i+1}$ inside $\mathcal{M}$, and this intersection is an embedded circle C' bounding a compact disk $B' \subset D_{\tau'}$. Since the symplectic area of B' is $T_{i+1} \leq T_i$, B_1 is not contained in the interior of B'. Hence, either ∂B_1 intersects $\partial B'$, or B_1 lies in $D_{\tau'} \setminus B'$, or ∂B_1 surrounds B'. In the first case, there exists a heteroclinic orbit from $P_{2,i}$ to $P_{2,i+1}$, which is a homoclinic orbit to $P_{2,i}$ if $P_{2,i} = P_{2,i+1}$. In the second and third cases, the forward flow of C intersects the cylinder connecting P_3^p and $P_{2,i+1}$ and thus also intersects the next family of disks $D_\tau, \tau \in (1, 2)$, bounding a new disk $B_2 \subset D_\tau$ with symplectic area T_i, for $\tau - 1 > 0$ sufficiently small. We may repeat the argument with the intersection between the forward flow of C and the local branch W_{i+2}^s of the stable manifold of $P_{2,i+2}$. Either it gives a heteroclinic orbit from to $P_{2,i}$ to $P_{2,i+2}$ or the forward flow of C intersects the rigid cylinder connecting P_3^p to $P_{2,i+2}$ on an embedded circle and then we can push it to the next family $D_\tau, \tau \in (2, 3)$. By contradiction, if the forward flow of C does not intersect any local branch of the stable manifold of any $P_{2,j}$ then it will intersect infinitely many times some rigid cylinder $V \in \mathcal{F}$ connecting P_3^p to some $P_{2,j}$. Denote by $C_n, n \in \mathbb{N}$, the embedded circles in V given by these intersections. We show that this leads to a contradiction since the symplectic area of V is finite. In fact, if C_n bounds a disk in V, then the symplectic area of such a disk is constant, equal to $T_i > 0$. By the uniqueness of solutions, such disks must be disjoint, and thus only finitely many such disks are allowed since the symplectic area of V is finite. If C_n is a non-trivial circle in V, then C_n and $P_{2,j}$ determine a half-cylinder in V with constant symplectic area $T_i - T_j > 0$. Hence, any two such embedded circles $C_n, C_m, m \neq n$, must intersect. This is not allowed by the uniqueness of solutions. We conclude that there exists at most a finite number of intersections between the forward flow of W_i^u with V. This implies that the forward flow of C eventually intersects the local branch of the stable manifold of some $P_{2,j}$, producing a heteroclinic orbit to if $j \neq i$ or a homoclinic orbit if $j = i$. This finishes the proof of Corollary 1.4.

4. The circular restricted three-body problem

The planar restricted three-body problem is the study of the dynamics of the following time-dependent Hamiltonian

$$E_\mu(q, p, t) = \frac{1}{2}|p|^2 - \frac{\mu}{|q - m(t)|} - \frac{1 - \mu}{|q - e(t)|}, \tag{4.1}$$

where $q = q_1 + iq_2 \in \mathbb{C} \equiv \mathbb{R}^2$ is the position of the massless satellite, $p = p_1 + ip_2$ is its momentum, $m(t)$ is the position of the moon with mass $0 < \mu < 1$, and $e(t)$ is the position of the earth with mass $1-\mu$. Hamilton's equations of E_μ are equivalent to

$$\ddot{q} = -\mu\frac{q-m(t)}{|q-m(t)|^3} - (1-\mu)\frac{q-e(t)}{|q-e(t)|^3}.$$

The moon and the earth move as in the Kepler problem. We fix their center of mass at 0, that is $(1-\mu)e(t)+\mu m(t) = 0, \forall t$, and assume that $e(t)$ and $m(t)$ move along ellipses with eccentricity $c \in [0,1]$. Assuming that $c=0$, we have

$$e(t) = -\mu e^{-it} \quad \text{and} \quad m(t) = (1-\mu)e^{-it}. \tag{4.2}$$

In the rotating system $\hat{q} = qe^{it}$, the earth and the moon are fixed at $-\mu$ and $1-\mu$ respectively. We obtain

$$\ddot{\hat{q}} = -\mu\frac{\hat{q}-(1-\mu)}{|\hat{q}-(1-\mu)|^3} - (1-\mu)\frac{\hat{q}+\mu}{|q+\mu|^3} + 2\dot{\hat{q}}i + \hat{q}. \tag{4.3}$$

Denoting $\hat{q}$ again by q, we see that the solutions of the autonomous Hamiltonian

$$H(q,p) := \frac{1}{2}|p|^2 - \frac{\mu}{|q-(1-\mu)|} - \frac{1-\mu}{|q+\mu|} + q_1p_2 - q_2p_1, \tag{4.4}$$

recover the trajectories of (4.3). Here, $1-\mu \in \mathbb{C}$ and $-\mu \in \mathbb{C}$ are the fixed positions of the moon and the earth in the rotation system, respectively. This system is referred to as the **circular planar restricted three-body problem**.

We briefly abbreviate the effective potential of H by

$$U(q) := U_0(q) - \frac{1}{2}|q|^2 = -\frac{\mu}{|q-(1-\mu)|} - \frac{1-\mu}{|q+\mu|} - \frac{1}{2}|q|^2,$$

where the dependence on μ in the notation is omitted. Hamilton's equations of H become

$$\begin{cases} \dot{q}_1 = p_1 - q_2, & \dot{p}_1 = -\partial_{q_1}U_0 - p_2 = -\partial_{q_1}U - p_2 - q_1, \\ \dot{q}_2 = p_2 + q_1, & \dot{p}_2 = -\partial_{q_2}U_0 + p_1 = -\partial_{q_2}U + p_1 - q_2, \end{cases} \tag{4.5}$$

or, equivalently,

$$\begin{cases} \ddot{q}_1 = -\partial_{q_1}U_0 - 2\dot{q}_2 + q_1 = -2\dot{q}_2 - \partial_{q_1}U, \\ \ddot{q}_2 = -\partial_{q_2}U_0 + 2\dot{q}_1 + q_2 = 2\dot{q}_1 - \partial_{q_2}U. \end{cases} \tag{4.6}$$

Notice that $F(q,\dot{q}) := |\dot{q}|^2/2 + U(q)$ is a conserved quantity of (4.6), and equals $H(q,p)$ under the correspondence $(q, \dot{q} = p+iq) \leftrightarrow (q,p)$. For any given $h \in \mathbb{R}$, the set $\Omega_h := \{U(q) \leq h\} \subset \mathbb{C}\backslash\{m,e\}$, is called the Hill region at energy h. The boundary $\partial\Omega_h$ is called the zero velocity curve since $\dot{q} = 0$ if $F(q,\dot{q}) = h$ and $q \in \partial\Omega_h$.

For any given $0 < \mu < 1$, the Hamiltonian H has five critical points $l_1, \ldots, l_5$, corresponding to equilibrium points of (4.5). The respective projections $\hat{l}_1, \hat{l}_2, \hat{l}_3$ of l_1, l_2 and l_3 to the q-plane are co-linear with e and m, and $\hat{l}_1$ lies in between the earth and the moon. If $0 < \mu < 1/2$, then $\hat{l}_2$ lies to the right of the moon and $\hat{l}_3$ to the left of the earth. The points l_1, l_2, l_3 are saddle-center equilibrium points of H, which means that the linearized vector field at those points admits a pair of real eigenvalues and a pair of purely imaginary eigenvalues. Writing $\hat{l}_1 = 1-\mu-r_1(\mu)$, we have

$$\frac{1-\mu}{(1-r_1)^2} - \frac{\mu}{r_1^2} = 1-\mu-r_1. \tag{4.7}$$

We can explicitly find the inverse function

$$\mu(r_1) = \frac{r_1^3(3-3r_1+r_1^2)}{1-2r_1+r_1^2+2r_1^3-r_1^4}, \tag{4.8}$$

and $r_1 \in (0,1)$ can be regarded as a parameter replacing μ. The projections $\hat{l}_4, \hat{l}_5$ of l_4, l_5 to the q-plane are symmetric with respect to the q_1-axis, and each one of them forms an equilateral triangle with the primaries.

Denote the critical values of $H = H_\mu$ by $L_i(\mu) := H_\mu(l_i),\ i = 1, \ldots, 5$. They satisfy $L_1 < L_2 \leq L_3 < L_4 = L_5$ and are called Lagrange values. The second inequality is an equality if and only if $\mu = 1/2$. We study the dynamics on $H^{-1}(E)$, where E is up to slightly above the first Lagrange

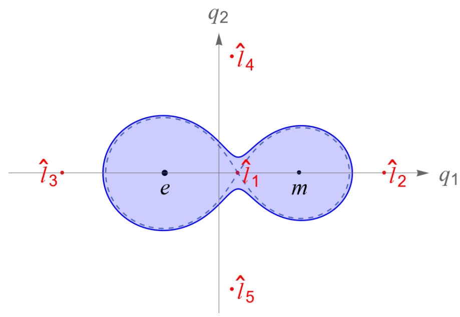

FIGURE 4.1. The projection of $\mathcal{M}^{e\#m}_{\mu,E}$ to the q-plane for E slightly above $L_1(\mu)$.

value $L_1(\mu)$. We are interested in the dynamics on the component of $H^{-1}(E)$ that projects to the the disk-like region with two punctures at the primaries, and has a small neck near $\hat{l}_1$, see Figure 4.1.

5. Proof of Theorem 1.5

In this section, we prove Theorem 1.5. We restate it for convenience.

Theorem 5.1. *Let $0 < \mu_0 < 1$. Then for every (μ, E) sufficiently close to $(\mu_0, L_1(\mu_0))$, with $E > L_1(\mu)$, the following statements hold:*

(i) *There exists a contact form $\alpha = \alpha_{\mu,E} = i_Y\omega_0$ on $\mathcal{M}^{e\#m}_{\mu,E} \equiv \mathbb{R}P^3\#\mathbb{R}P^3$ whose Reeb flow is equivalent to the regularized Hamiltonian flow. Here, $Y = Y_{\mu,E}$ is a Liouville vector field, defined on a neighborhood of $\mathcal{M}^{e\#m}_{\mu,E}$ in $\mathbb{R}^4$ and transverse to $\mathcal{M}^{e\#m}_{\mu,E}$. Also, ω_0 is the canonical symplectic form $\sum_i dp_i \wedge dq_i$.*

(ii) *There exists a compatible almost complex structure $J = J_{\mu,E}$ on $\mathbb{R} \times \mathcal{M}^{e\#m}_{\mu,E}$ adapted to α that admits a pair of J-holomorphic planes asymptotic to $P_{2,E}$ through opposite directions. The closure of their projections to $\mathcal{M}^{e\#m}_{\mu,E}$ form a regular two-sphere $\mathcal{S} = \mathcal{S}_{\mu,E}$ containing $P_{2,E}$. Furthermore, dist$(\mathcal{S}, l_1(\mu)) \to 0$ as $E \to L_1(\mu)^+$ uniformly in μ.*

(iii) *There exists a contact form $\alpha = \alpha_{\mu,L_1(\mu)} = i_{Y_{\mu,L_1(\mu)}}\omega_0$ on the sphere-like singular subset $\dot{\mathcal{M}}^e := \mathcal{M}^e_{\mu,L_1(\mu)} \setminus \{l_1(\mu)\}$ so that the contact forms $\alpha_{\mu,E}$ in (i) converge in $C^\infty_{loc}(\dot{\mathcal{M}}^e)$ to $\alpha_{\mu,L_1(\mu)}$ as $E \to L_1(\mu)^+$ uniformly in μ. The same conclusion holds for $\dot{\mathcal{M}}^m := \mathcal{M}^m_{\mu,L_1(\mu)} \setminus \{l_1(\mu)\}$.*

The proof has two main steps. In the first step, we study the limiting linear system near $l_1(\mu_0)$ for every (μ, E), $E > L_1(\mu)$, sufficiently close to $(\mu_0, L_1(\mu_0))$. Under a suitable re-scaling of variables, the limiting system becomes a standard linear system with a saddle-center equilibrium at the origin. It thus admits a natural Liouville vector field Y_2. For a suitable choice of almost complex structure J, a pair of J-holomorphic planes can be explicitly found, which are asymptotic to the Lyapunoff orbits through opposite directions and form a spherical shield. Finally, the automatic transversality of such J-holomorphic planes, see Theorem 2.6, guarantees the existence of similar J-holomorphic planes for every (μ, E), $E > L_1(\mu)$, sufficiently close to $(\mu_0, L_1(\mu_0))$. In particular, this planes project arbitrarily close to $l_1(\mu_0)$. The second step is to construct a suitable interpolation between Y_2 and the natural Liouville vector fields Y_e, Y_m centered at the earth and the moon, see in [1].

5.1. **Spherical shields near the neck.** In this section, we construct a pair of J-holomorphic planes $\tilde{u}_{1,E} = (a_{1,E}, u_{1,E}), \tilde{u}_{2,E} = (a_{2,E}, u_{2,E}) : \mathbb{C} \to \mathbb{R} \times H^{-1}(E)$ asymptotic to the Lyapunoff orbit $P_{2,E}$ near $l_1(\mu)$, for (μ, E) sufficiently close to $(\mu_0, L_1(\mu_0))$, with $E > L_1(\mu)$. Recall that $H(p,q) = \frac{1}{2}((p_1 - q_2)^2 + (p_2 + q_1)^2) + U(q)$, where

$$U(q) = -\frac{\mu}{|q - (1-\mu)|} - \frac{1-\mu}{|q + \mu|} - \frac{1}{2}|q|^2.$$

The energy of the first Lagrange point is denoted

$$L_1(\mu) = U(\hat{l}_1(\mu)) = -\frac{\mu}{r_1} - \frac{1-\mu}{1-r_1} - \frac{1}{2}(1-\mu-r_1)^2$$

where $\hat{l}_1(\mu) = (1-\mu-r_1, 0)$ denotes the projection of $l_1(\mu)$ to the q-plane and $r_1 \in (0,1)$ solves (4.7). Given $\delta, \epsilon_0, c_0 > 0$, we define $\hat{q}_1^{\pm} := 1-\mu-r_1 \pm \epsilon^{1/2}\delta$ and

$$\hat{\mathcal{L}}_\epsilon := \left\{(p,q) \in \mathcal{M}^{e\#m}_{\mu, L_1(\mu)+c_0\epsilon},\ q_1 \in [\hat{q}_1^-, \hat{q}_1^+]\right\}. \tag{5.1}$$

Let $(\bar{p}, \bar{q}) = (p,q) - l_1(\mu) = (p + i\hat{l}_1(\mu), q - \hat{l}_1(\mu))$ be shifted coordinates. The Hamiltonian H near $l_1(\mu)$ can be written as

$$H(\bar{p}, \bar{q}) = L_1(\mu) + H_2(\bar{p}, \bar{q}) + R(\bar{p}, \bar{q}),$$

where $H_2(\bar{p}, \bar{q}) = \frac{1}{2}\big((\bar{p}_1 - \bar{q}_2)^2 + (\bar{p}_2 + \bar{q}_1)^2 + (\nabla^2 U(\hat{l}_1)\bar{q}, \bar{q})\big)$, R vanishes up to order 2 at $l_1(\mu)$,

$$\nabla^2 U(\hat{l}_1) = \begin{pmatrix} -1-4a & 0 \\ 0 & -1+2a \end{pmatrix}, \quad a = \frac{1-\mu}{2(1-r_1)^3} + \frac{\mu}{2r_1^3}.$$

Using (4.7), we compute

$$a(r_1) = \frac{2 - r_1 + r_1^2}{1 - 2r_1 + r_1^2 + 2r_1^3 - r_1^4}. \tag{5.2}$$

Moreover, $\max\{a(r_1), r_1 \in [0,1]\} = a(1/2) = 4$ and $\min\{a(r_1), r_1 \in [0,1]\} = a(0) = a(1) = 2$ as one readily checks. We re-scale $(\bar{p}, \bar{q})$ and obtain a new Hamiltonian in coordinates $(\hat{p}, \hat{q})$

$$H_\epsilon(\hat{p}, \hat{q}) := \epsilon^{-1}(H(\epsilon^{1/2}\hat{p} - i(1-\mu-r_1), \epsilon^{1/2}\hat{q} + (1-\mu-r_1)) - L_1(\mu)).$$

Notice that $H_\epsilon(\hat{p}, \hat{q})$ converges in C^∞_{loc} to $H_2(\hat{p}, \hat{q})$ as $\epsilon \to 0^+$. In coordinates $(\hat{p}, \hat{q})$, the energy surface $H^{-1}(L_1(\mu) + c_0\epsilon)$ and the re-scaled region $\hat{\mathcal{L}}_\epsilon$ smoothly converge to $H_2^{-1}(c_0)$ and $H_2^{-1}(c_0) \cap \{\hat{q}_1 \in [-\delta, \delta]\}$, respectively.

The dynamics of H_2 is determined by the linear system

$$\begin{pmatrix} \dot{\hat{p}} \\ \dot{\hat{q}} \end{pmatrix} = J_4 \nabla^2 H(l_1(\mu)) \begin{pmatrix} \hat{p} \\ \hat{q} \end{pmatrix} = \begin{pmatrix} J_2 & -\nabla^2 U(\hat{l}_1(\mu)) - I_2 \\ I_2 & J_2 \end{pmatrix} \begin{pmatrix} \hat{p} \\ \hat{q} \end{pmatrix},$$

where $\lambda_1, \lambda_2 > 0$ satisfy

$$\nabla^2 H(l_1(\mu)) = \begin{pmatrix} I_2 & J_2 \\ J_2^T & \operatorname{diag}(-4a, 2a) \end{pmatrix}.$$

Here, $J_{2n} = \begin{pmatrix} 0_n & -I_n \\ I_n & 0_n \end{pmatrix}$ is the standard complex matrix, and 0_n and I_n are the $n \times n$ zero and identity matrices, respectively. The eigenvalues of $J_4 \nabla^2 H(l_1(\mu))$ are $\pm\lambda_1$ and $\pm i\lambda_2$, where

$$\lambda_1^2 = a - 1 + \sqrt{a(9a-4)} \geq 1 + 2\sqrt{7} \quad \text{and} \quad \lambda_2^2 = 1 - a + \sqrt{a(9a-4)} \geq -1 + 2\sqrt{7}. \tag{5.3}$$

Notice that $\lambda_1 \geq \lambda_2$ since $a \geq 2$. The following vectors form a symplectic basis of $\mathbb{R}^4$

$$\begin{aligned}
V_1 &:= k_1 \left(\frac{5a + \sqrt{a(9a-4)}}{2(1+4a)\sqrt{a(9a-4)}}, 0, 0, \frac{2 + 3a - \sqrt{a(9a-4)}}{2(1+4a)\sqrt{a(9a-4)}} \right), \\
V_2 &:= k_2 \left(0, \frac{3a}{2\sqrt{a(9a-4)}} + \frac{1}{2}, \frac{1}{\sqrt{a(9a-4)}}, 0 \right), \\
V_3 &:= k_1^{-1} \left(0, \frac{(1+4a)(3a - \sqrt{a(9a-4)})}{2 + 3a - \sqrt{a(9a-4)}}, \frac{2(1+4a)}{2 + 3a - \sqrt{a(9a-4)}}, 0 \right), \\
V_4 &:= k_2^{-1} \left(\frac{5a - \sqrt{a(9a-4)}}{2 + 3a + \sqrt{a(9a-4)}}, 0, 0, 1 \right),
\end{aligned} \tag{5.4}$$

where

$$k_1 = \frac{\sqrt{2}(a(9a-4))^{1/4}(1+4a)}{(2+3a-\sqrt{a(9a-4)})^{1/2}(-1+a+\sqrt{a(9a-4)})^{1/4}},$$
$$k_2 = \frac{\sqrt{2}(a(9a-4))^{1/4}(1-a+\sqrt{a(9a-4)})^{1/4}}{(2+3a+\sqrt{a(9a-4)})^{1/2}}.$$

Moreover, $\mathbb{R}V_1 \oplus \mathbb{R}V_3$ contains the eigenspaces of $\pm\lambda_1$ and $\mathbb{R}V_2 \oplus \mathbb{R}V_4$ contains the generalized eigenspace of $\pm i\lambda_2$. We write $V := \left(V_1^T, V_2^T, V_3^T, V_4^T\right) \in \mathrm{Sp}(4)$. Let

$$C_0 := \sqrt{2}(a(9a-4))^{1/4}, \quad C_1 := (2+3a-(a(9a-4))^{1/2})^{1/2}, \tag{5.5}$$
$$C_2 := (2+3a+(a(9a-4))^{1/2})^{1/2}.$$

Then V gets simplified as

$$V = \begin{pmatrix} \frac{C_2^2+(2a-2)}{\sqrt{\lambda_1}C_1C_0} & 0 & 0 & \frac{C_1^2+(2a-2)}{\sqrt{\lambda_2}C_2C_0} \\ 0 & \frac{\sqrt{\lambda_2}(C_2^2-2)}{C_2C_0} & \frac{\sqrt{\lambda_1}(C_1^2-2)}{C_1C_0} & 0 \\ 0 & \frac{2\sqrt{\lambda_2}}{C_2C_0} & \frac{2\sqrt{\lambda_1}}{C_1C_0} & 0 \\ \frac{C_1}{\sqrt{\lambda_1}C_0} & 0 & 0 & \frac{C_2}{\sqrt{\lambda_2}C_0} \end{pmatrix}. \tag{5.6}$$

Under the linear symplectic change of coordinates $\Phi(x) := Vx^T = (\hat{p}, \hat{q})^T$, H_2 becomes

$$H_2 = \frac{\lambda_1}{2}(x_1^2 - x_3^2) + \frac{\lambda_2}{2}(x_2^2 + x_4^2), \tag{5.7}$$

and thus Hamilton's equations decouple

$$\begin{cases} \dot{x}_1 = \lambda_1 x_3, \\ \dot{x}_3 = \lambda_1 x_1, \end{cases} \quad \text{and} \quad \begin{cases} \dot{x}_2 = -\lambda_2 x_4, \\ \dot{x}_4 = \lambda_2 x_2. \end{cases} \tag{5.8}$$

The energy surface $H_2^{-1}(c_0)$, $c_0 > 0$, contains a unique Lyapunov orbit

$$P_{0,L} = \left(0, \sqrt{2c_0/\lambda_2}\cos(\lambda_2 t), 0, \sqrt{2c_0/\lambda_2}\sin(\lambda_2 t)\right), \quad t \in \mathbb{R}/\frac{2\pi}{\lambda_2}\mathbb{Z}, \tag{5.9}$$

which is hyperbolic and has index 2.

Let $\Lambda := \mathrm{diag}(1-b, 1/2, b, 1/2)$, where $0 < b < 1$. We consider the Liouville vector field

$$Y_2 := \Lambda \cdot x\partial_x = (1-b)x_1\partial_{x_1} + \frac{1}{2}x_2\partial_{x_2} + bx_3\partial_{x_3} + \frac{1}{2}x_4\partial_{x_4}. \tag{5.10}$$

We compute on $H_2^{-1}(c_0)$

$$dH_2 \cdot Y_2 = \lambda_1(1-b)x_1^2 - \lambda_1 b x_3^2 + \frac{\lambda_2}{2}(x_2^2 + x_4^2).$$

Hence, Y_2 is transverse to $H_2^{-1}(c_0), c_0 > 0$, if $|x_3|$ is sufficiently small. Notice that $x_1 = x_3 = 0$ implies $x_2^2 + x_4^2 > 0$. Under this condition, Y_2 induces a contact form α_2 on $H_2^{-1}(c_0)$ near $x_3 = 0$, given by the restriction of

$$\alpha_2 := \iota_{Y_2}\omega = -bx_3dx_1 - \frac{1}{2}x_4dx_2 + (1-b)x_1dx_3 + \frac{1}{2}x_2dx_4.$$

The contact structure $\xi = \ker\alpha_2$ is spanned by $\xi_1 := X_1 - A_1Y_2$ and $\xi_2 := X_2 - A_2Y_2$, where

$$A_1 = \frac{dH_2 \cdot X_1}{dH_2 \cdot Y_2}, \quad A_2 = \frac{dH_2 \cdot X_2}{dH_2 \cdot Y_2},$$
$$X_1 = \frac{1}{2}x_4\partial_{x_1} - bx_3\partial_{x_2} + \frac{1}{2}x_2\partial_{x_3} - (1-b)x_1\partial_{x_4},$$
$$X_2 = -\frac{1}{2}x_2\partial_{x_1} + (1-b)x_1\partial_{x_2} + \frac{1}{2}x_4\partial_{x_3} - bx_3\partial_{x_4}.$$

The Reeb vector field of α_2 is $R_2 = (dH_2(Y_2))^{-1}X_{H_2}$. We choose the almost complex structure J on $\mathbb{R} \times H_2^{-1}(c_0)$ such that $J \cdot \xi_1 = \xi_2$ and $J \cdot \partial_a = R_2$.

Next we construct a J-holomorphic cylinder $\tilde{u} = (a, u) : \mathbb{R} \times \mathbb{R}/\mathbb{Z} \to \mathbb{R} \times H_2^{-1}(c_0)$, asymptotic to $P_{0,L}$ at $+\infty$, and with a removable singularity at $-\infty$. We have $\pi\partial_t u = J \cdot \pi\partial_s u$, $\partial_s a = \alpha_2(\partial_t u)$, and $\partial_t a = -\alpha_2(\partial_s u)$, where $\pi : TH_2^{-1}(c_0) \to \xi$ is the projection along R_2. In coordinates $a \in \mathbb{R}$ and $(x_1, x_2, x_3, x_4) \in \mathbb{R}^4$, we assume that

$$a(s,t) = a(s) \quad \text{and} \quad u(s,t) = (x_1(s), r(s)\cos(2\pi t), 0, r(s)\sin(2\pi t)),$$

where $x_1(s)^2 = (2c_0 - \lambda_2 r(s)^2)/\lambda_1 > 0$ and $r(s) > 0$. An extensive computation gives

$$r'(s) = \frac{-2\pi r(\lambda_2 r^2 - 2c_0)(\lambda_1 r^2 + 4(b-1)^2(2c_0 - \lambda_2 r^2))}{(2c_0 + (1-2b)(2c_0 - \lambda_2 r^2))^2} =: g(r).$$

Since $0 < b < 1$ and $0 < r < r_0 := \sqrt{2c_0/\lambda_2}$, we have $2c_0 + (1-2b)(2c_0 - \lambda_2 r^2) > 0$, and thus $g = g(r)$ is well-defined on the interval $[-r_0, r_0]$. It vanishes at $r = 0, \pm r_0$, is positive on $(0, r_0)$ and negative on $(-r_0, 0)$. Assume that $r(0) \in (0, r_0)$ and $a(0) = 0$. Then $r(s) \to r_0$ exponentially fast as $s \to +\infty$. We also have $a'(s) = \alpha_2(\partial_t u) = \pi r(s)^2 \to \pi r_0^2 = 2\pi c_0/\lambda_2$ as $s \to +\infty$. Hence $s = +\infty$ is a positive end asymptotic to $P_{0,L}$. Notice that $x_1(s) \to 0$ as $s \to +\infty$. The end at $s = -\infty$ is removable since $r(s), a'(s) \to 0$ and $x_1(s) \to \sqrt{2c_0/\lambda_1}$ as $s \to -\infty$. In particular, $u(s,t) \to \left(\sqrt{2c_0/\lambda_1}, 0, 0, 0\right)$ as $s \to -\infty$. Hence, we obtain a J-holomorphic plane $\tilde{u}_1 : \mathbb{C} \to \mathbb{R} \times H_2^{-1}(c_0)$ asymptotic to $P_{0,L}$. Its energy is $E(\tilde{u}_1) = \pi r_0^2 = 2\pi c_0/\lambda_2$ and coincides with the action of $P_{0,L}$. Similarly, if $r(0) \in (-r_0, 0)$, then we obtain another J-holomorphic plane $\tilde{u}_2 : \mathbb{C} \to \mathbb{R} \times H_2^{-1}(c_0)$ asymptotic to $P_{0,L}$. Both J-holomorphic planes and their projections to $H_2^{-1}(c_0) \cap \{x_3 = 0\}$ are embedded and do not intersect each other. They approach $P_{0,L}$ through opposite directions. By uniqueness of such J-holomorphic planes, these are the only J-holomorphic planes asymptotic to $P_{0,L}$ up to reparametrization and $\mathbb{R}$-translation, see Theorem 2.3. The discussion above implies the following proposition.

Proposition 5.2. *Let $0 < b < 1$ and $c_0 > 0$. Then the following assertions hold on $H_2^{-1}(c_0)$:*

(i) *There exists a neighborhood U_2 of $H_2^{-1}(c_0) \cap \{x_3 = 0\}$ in $H_2^{-1}(c_0)$, so that Y_2 is transverse to U_2. Let $\xi := \ker \alpha_2$, where $\alpha_2 := \iota_{Y_2}\omega|_{U_2}$ is the contact form on U_2 induced by Y_2, and let R_2 be the Reeb vector field of α_2.*
(ii) *There exists a compatible $\mathbb{R}$-invariant almost complex structure J on $\mathbb{R} \times H_2^{-1}(c_0)$ satisfying $J \cdot \xi = \xi$ and $J \cdot \partial_a = R_2$, admitting a pair of embedded holomorphic planes $\tilde{u}_1 = (a_1, u_1), \tilde{u}_2 = (a_2, u_2) : \mathbb{C} \to \mathbb{R} \times H_2^{-1}(c_0)$, asymptotic to $P_{0,L}$ through opposite directions, and satisfying $u_1(\mathbb{C}) \cup u_2(\mathbb{C}) = H_2^{-1}(c_0) \cap \{x_3 = 0\}$.*

We fix $0 < \mu_0 < 1$, $c_0 = 1$ and $0 < b < 1$. The re-scaled energy surface $H_\epsilon^{-1}(1), \epsilon > 0$, near $l_1(\mu)$, converges in C^∞_{loc} to $H_2^{-1}(1)$ as $(\mu, \epsilon) \to (\mu_0, 0^+)$. Since $P_{0,L}$ is hyperbolic, $H_\epsilon^{-1}(1)$ admits an index-2 hyperbolic orbit $P_{2,\epsilon}$ converging in C^∞ to $P_{0,L}$ as $(\mu, \epsilon) \to (\mu_0, 0^+)$. We may assume that there exists a small compact tubular neighborhood $\mathcal{U} \subset \mathbb{R}^4$ of $H_2^{-1}(1) \cap \{-\delta \leq x_3 \leq \delta\}$, for some $\delta > 0$ small, so that Y_2 is transverse to $H_\epsilon^{-1}(1) \cap \mathcal{U}$ for every $(\mu, \epsilon), \epsilon > 0$, sufficiently close to $(\mu_0, 0)$. Denote by $\alpha_\epsilon = i_{Y_2}\omega_0|_{H_\epsilon^{-1}(1) \cap \mathcal{U}}$ the contact form on the neck-region $H_\epsilon^{-1}(1) \cap \mathcal{U}$ induced by Y_2. Up to a diffeomorphism C^∞-close to the identity, we can see α_ϵ as a contact form on $H_2^{-1}(1) \cap \mathcal{U}$ so that $\alpha_\epsilon \to \alpha_2$ in C^∞_{loc} as $(\mu, \epsilon) \to (\mu_0, 0^+)$. We may choose any almost complex structure $J_\epsilon = J_{\mu,\epsilon}$ on $\mathbb{R} \times (H_2^{-1}(1) \cap \mathcal{U})$ adapted to α_ϵ so that $J_\epsilon \to J$ in C^∞_{loc} as $(\mu, \epsilon) \to (\mu_0, 0^+)$. Since the curves $\tilde{u}_1, \tilde{u}_2$ are Fredholm regular, see Proposition 2.6-(iii), we find for each $(\mu, \epsilon), \epsilon > 0$ sufficiently close to $(\mu_0, 0)$, a pair of J_ϵ-holomorphic planes $\tilde{u}_{1,\epsilon} = (a_{1,\epsilon}, u_{1,\epsilon})$, $\tilde{u}_{2,\epsilon} = (a_{2,\epsilon}, u_{2,\epsilon}) : \mathbb{C} \to \mathbb{R} \times H_\epsilon^{-1}(1)$ asymptotic to $P_{2,\epsilon}$. Recall that $H_\epsilon = \epsilon^{-1}(H \circ \phi - L_1(\mu))$, where $\phi : (\hat{p}, \hat{q}) \mapsto (p, q) = \epsilon^{1/2}(\hat{p}, \hat{q}) + l_1(\mu)$ induces a diffeomorphism between suitable subsets of $H_\epsilon^{-1}(1)$ and $H^{-1}(L_1(\mu) + \epsilon)$. We obtain $\hat{J}_\epsilon$-holomorphic curves $\tilde{U}_{i,\epsilon} = (A_{i,\epsilon}, U_{i,\epsilon}) : \mathbb{C} \to \mathbb{R} \times H^{-1}(L_1(\mu) + \epsilon)$, $i = 1, 2$, given by

$$\tilde{U}_{i,\epsilon}(s,t) := (\epsilon a_{i,\epsilon}(s,t), \phi(u_{i,\epsilon}(s,t))) = (\epsilon a_{i,\epsilon}(s,t), \epsilon^{1/2} u_{i,\epsilon}(s,t) + l_1(\mu)), \quad \forall (s,t) \in \mathbb{R} \times \mathbb{R}/\mathbb{Z}.$$

Here, the almost complex structure $\hat{J}_\epsilon$ on $\mathbb{R} \times H^{-1}(L_1(\mu)+\epsilon)$ is the one induced by the contact form $\hat{\alpha}_\epsilon := \epsilon(\phi^{-1})^*\alpha_\epsilon$, and the push-forward $\phi_* J_\epsilon|_{\xi=\ker \alpha_\epsilon}$. Indeed, we check that

$$\begin{aligned} \partial_s A_{i,\epsilon} &= \epsilon \partial_s a_{i,\epsilon} = \epsilon \alpha_\epsilon(\partial_t u_{i,\epsilon}) = \epsilon \alpha_\epsilon (D\phi^{-1}(U_{i,\epsilon}) \cdot \partial_t U_{i,\epsilon}) \\ &= \epsilon(\phi^{-1})^* \alpha_\epsilon(\partial_t U_{i,\epsilon}) = \hat{\alpha}_\epsilon(\partial_t U_{i,\epsilon}), \\ \pi(\partial_s U_{i,\epsilon}) &= D\phi(u_{i,\epsilon})\pi(\partial_s u_{i,\epsilon}) = D\phi(u_{i,\epsilon}) J_\epsilon \cdot \pi(-\partial_t u_{i,\epsilon}) \\ &= (D\phi J_\epsilon D\phi^{-1})|_{U_{i,\epsilon}} \cdot \pi(-\partial_t U_{i,\epsilon}) = (\phi_* J_\epsilon)|_{U_{i,\epsilon}} \cdot \pi(-\partial_t U_{i,\epsilon}). \end{aligned}$$

Denote $\omega = d\hat{p} \wedge d\hat{q}$ the standard symplectic form. Since $\phi_* Y_2 = Y_2$ and $\phi^*\omega = \epsilon\omega$, we have $\hat{\alpha}_\epsilon = \iota_{Y_2}\omega|_{H^{-1}(L_1(\mu)+\epsilon)}$. As observed before, $\hat{\alpha}_\epsilon$ restricts to a contact form on $H^{-1}(L_1(\mu)+\epsilon)$ near $x_3 = 0$. We have proved the following proposition.

Proposition 5.3. *Fix $0 < \mu_0 < 1$. Then for every $(\mu, \epsilon), \epsilon > 0$, sufficiently close to $(\mu_0, 0)$, the following assertions hold:*

(i) *There exists a contact form $\hat{\alpha}_\epsilon = \hat{\alpha}_{\mu,\epsilon}$ on a small neighborhood U_ϵ of $\{x_3 = 0\}$ in $H^{-1}(L_1(\mu)+\epsilon)$ so that its Reeb vector field is parallel to the Hamiltonian vector field of H.*

(ii) *There exists an almost complex structure $\hat{J}_\epsilon = \hat{J}_{\mu,\epsilon}$ on $\mathbb{R} \times H^{-1}(L_1(\mu)+\epsilon)$ near $x_3 = 0$, adapted to $\hat{\alpha}_\epsilon$, and a pair of $\hat{J}_\epsilon$-holomorphic planes $\tilde{U}_{i,\epsilon} = (A_{i,\epsilon}, U_{i,\epsilon}) : \mathbb{C} \to \mathbb{R} \times H^{-1}(L_1(\mu)+\epsilon), i = 1, 2,$ asymptotic to the Lyapunoff orbit $P_{2,\epsilon} \subset H^{-1}(L_1(\mu)+\epsilon)$ through opposite directions.*

(iii) *Given any neighborhood $\mathcal{U} \subset \mathbb{R}^4$ of $l_1(\mu_0)$, we have $U_{1,\epsilon}(\mathbb{C}) \cup U_{2,\epsilon}(\mathbb{C}) \subset \mathcal{U}$ for every $(\mu, \epsilon), \epsilon > 0$ sufficiently close to $(\mu_0, 0)$.*

5.2. **Interpolation of Liouville vector fields.** Fix $0 < \mu_0 < 1$. For every (μ, E) sufficiently close to $(\mu_0, L_1(\mu_0))$, we denote by $\mathcal{D}_\mu \subset \mathbb{C}$ the closed disk of radius $0 < \hat{l}_1(\mu) + \mu < 1$ centered at the earth $q = -\mu$. Notice that $\hat{l}_1(\mu) \in \partial\mathcal{D}_\mu = C_\mu := \{(q_1+\mu)^2 + q_2^2 = (\hat{l}_1(\mu)+\mu)^2\}$. Recall that $B_\mu \subset \mathbb{C} \setminus \{-\mu\}$ is the projection of $\mathcal{M}^e_{\mu, L_1(\mu)}$ to the q-plane. Consider the Liouville vector field $Y_e := (q_1+\mu)\partial_{q_1} + q_2\partial_{q_2}$ in coordinates (p,q) centered at the earth. We also consider the Liouville vector field $Y_m := (q_1 - (1-\mu))\partial_{q_1} + q_2\partial_{q_2}$ centered at the moon. The following proposition is essentially proved in [1] with a slightly different notation.

Proposition 5.4 (Albers-Frauenfelder-van Koert-Paternain [1])**.** *Consider polar coordinates $q = (-\mu + \rho\cos\theta, \rho\sin\theta), \rho > 0, \theta \in \mathbb{R}/2\pi\mathbb{Z}$, centered at the earth $q = -\mu$. Then*

(i) *$B_\mu \setminus \{\hat{l}_1(\mu)\} \subset \mathcal{D}_\mu \setminus \partial\mathcal{D}_\mu$ and $\partial_\rho U > 0$ on $\mathcal{D}_\mu \setminus \{-\mu, \hat{l}_1(\mu)\}$.*

(ii) *For every $0 < \rho < 1$, the even function $\theta \mapsto U(-\mu + \rho\cos\theta, \rho\sin\theta)$ attains its minimum at $\theta = 0$ ($\theta = \pi$ is a local minimum), and its maximum at $\pm\hat{\theta}$, where $\hat{\theta} \in (0,\pi)$ satisfies $2\cos\hat{\theta} = \rho$, i.e., $\hat{q} := (-\mu + \rho\cos\hat{\theta}, \rho\sin\hat{\theta})$ is such that $|\hat{q} - (1-\mu)| = 1$. Moreover, $\theta \mapsto U(-\mu + \rho\cos\theta, \rho\sin\theta)$ is strictly increasing on $[0, \hat{\theta}]$ and strictly decreasing on $[\hat{\theta}, \pi]$.*

(iii) *For every $E < L_1(\mu)$, Y_e is positively transverse to the unregularized component $\mathcal{M}^e_{\mu,E} \subset H^{-1}(E)$, i.e. $dH \cdot Y_e > 0$ on $\mathcal{M}^e_{\mu,E}$.*

(iv) *Let $E \geq L_1(\mu)$ and $\theta \in \mathbb{R}/2\pi\mathbb{Z}$. If there exists $\rho_\theta \in (0, \hat{l}_1(\mu)+\mu)$ such that $U(-\mu + \rho_\theta\cos\theta, \rho_\theta\sin\theta) = E$, then $dH \cdot Y_e > 0$ on the unregularized component $\mathcal{M}^e_{\mu,E} \cap \{(p,q), q = (-\mu + \rho\cos\theta, \rho\sin\theta), 0 < \rho \leq \rho_\theta\}$.*

Remark 5.5. *An analogous statement holds on the moon side via the symmetry $H_\mu(p,q) = H_{1-\mu}(-p,-q)$.*

Proof. Item (i) follows from Corollary 5.3 and Lemma 5.4 in [1], (ii) follows from Lemma 5.2 in [1], (iii) follows from Proposition 5.1 in [1], and (iv) follows from the proof of Proposition 5.1 in [1]. □

Let $\gamma(\theta) := (-\mu + (\hat{l}_1(\mu)+\mu)\cos\theta, (\hat{l}_1(\mu)+\mu)\sin\theta)), \theta \in \mathbb{R}/2\pi\mathbb{Z}$, be a parametrization of C_μ. Then $\gamma(0) = \hat{l}_1(\mu)$. By Proposition 5.4, the function $\theta \mapsto U(\gamma(\theta))$ attends its minimum

$L_1(\mu)$ precisely at $\theta = 0$. Hence, for every $\epsilon := E - L_1(\mu) > 0$ sufficiently small, there exists $\theta_* = \theta_*(\epsilon) > 0$ arbitrarily small so that

$$\begin{aligned} &U(\gamma(\pm\theta_*)) = L_1(\mu) + \epsilon, \\ &U(\gamma(\theta)) > L_1(\mu) + \epsilon, \quad \forall \theta_* < \theta < 2\pi - \theta_*, \\ &U(\gamma(\theta)) < L_1(\mu) + \epsilon, \quad \forall - \theta_* < \theta < \theta_*. \end{aligned}$$

Proposition 5.6. *Let $\mathcal{D}_\epsilon \subset \mathcal{D}_\mu$ be the open set $\mathcal{D}_\mu \setminus (\{-\mu\} \cup \partial\mathcal{D}_\mu)$ with the annular sector $(\rho, \theta) \in [\mu + \hat{l}_1(\mu) - \epsilon^{1/2}, \mu + \hat{l}_1(\mu)] \times [-\theta_*, \theta_*]$ removed. For every $\epsilon > 0$ sufficiently small, $dH \cdot Y_e > 0$ on $H^{-1}(L_1(\mu) + \epsilon) \cap \{(p, q), q \in \mathcal{D}_\epsilon\}$.*

Proof. Let $\epsilon > 0$ be small. Since $U(\gamma(\theta)) > L_1(\mu) + \epsilon$, for every $\theta_* < \theta < 2\pi - \theta_*$, Proposition 5.4-(i) and (iv) implies that $dH \cdot Y_e > 0$ on $H^{-1}(L_1(\mu)+\epsilon) \cap \{(p, q), q = (-\mu + \rho\cos\theta, \rho\sin\theta), 0 < \rho < \hat{l}_1(\mu) + \mu, \theta_* < \theta < 2\pi - \theta_*\}$. Hence, it remains to check that $dH \cdot Y_e > 0$ on $H^{-1}(L_1(\mu)+\epsilon) \cap \{0 < \rho < \mu + \hat{l}_1(\mu) - \epsilon^{1/2}, -\theta_* < \theta < \theta_*\}$.

Assume that $\theta \in [-\theta_*, \theta_*]$, and denote $p = \dot{q} - iq = (r_{\dot{q}} \cos\theta_{\dot{q}} + \rho\sin\theta, r_{\dot{q}} \sin\theta_{\dot{q}} + \mu - \rho\cos\theta)$. Then

$$\begin{aligned} dH \cdot Y_e &= (p_2 + q_1 + \partial_{q_1} U)(q_1 + \mu) + (q_2 - p_1 + \partial_{q_2} U)q_2 \\ &= \rho(\partial_\rho U - r_{\dot{q}} \sin(\theta - \theta_{\dot{q}})) \geq \rho(\partial_\rho U - r_{\dot{q}}) \\ &= \frac{1-\mu}{\rho} + \frac{\mu(\rho - \cos\theta)\rho}{(1 + \rho^2 - 2\rho\cos\theta)^{3/2}} - \rho^2 + \mu\rho\cos\theta - \rho r_{\dot{q}}, \end{aligned}$$

where $r_{\dot{q}}^2 = 2(L_1(\mu) + \epsilon - U)$. For simplicity, we denote $s = \cos\theta$. Let

$$L(\rho, s) := \frac{1-\mu}{\rho} + \frac{\mu(\rho - s)\rho}{(1 + \rho^2 - 2\rho s)^{3/2}} - \rho^2 + \mu\rho s - \rho\sqrt{2(\epsilon + L_1(\mu) - U)}.$$

We compute

$$\frac{1}{\mu\rho}\partial_s L = \frac{3\rho(\rho - s)}{(1 + \rho^2 - 2\rho s)^{5/2}} + 1 - \frac{1}{(1 + \rho^2 - 2\rho s)^{3/2}} - \frac{\rho(1 + \rho^2 - 2\rho s)^{-3/2} - \rho}{\sqrt{2(\epsilon + L_1(\mu) - U)}}.$$

Recall that $\theta \in [-\theta_*, \theta_*]$. For $\epsilon > 0$ sufficiently small, $\theta_* > 0$ is arbitrarily close to 0 and thus $s = \cos\theta$ is arbitrarily close to 1. This implies that $1 + \rho^2 - 2\rho s = 1 - \rho(2s - \rho) \in (0, 1)$ for every $\rho \in (0, 1)$. Moreover, if $\epsilon > 0$ is sufficiently small, we have $\rho < s = \cos\theta$ for every $\rho < 1 - r_1 = \mu + \hat{l}_1(\mu) < 1$. We conclude that if $\epsilon > 0$ is sufficiently small and $\theta \in [-\theta_*, \theta_*]$, then $\partial_s L < 0$. This implies that L is strictly increasing in $\theta \in [0, \theta_*]$, if $\epsilon > 0$ is sufficiently small and $0 < \rho < 1 - r_1$. By symmetry, it suffices to check that $L > 0$ when $s = 1$ (or $\theta = 0$) and $\rho \in (0, 1 - r_1 - \epsilon^{1/2})$.

We fix $s = 1$. Notice that $L/\rho = \partial_\rho U - \sqrt{2(L_1(\mu) + \epsilon - U)}$. We know that $\partial_\rho U > 0$ if $\rho \in (0, 1 - r_1)$. Hence $L > 0$ if and only if $F_1(\rho) := (\partial_\rho U)^2 - 2(L_1(\mu) + \epsilon - U) > 0$. After a straightforward computation using the expression (5.2) for $\mu = \mu(r_1)$, we obtain

$$F_1'(\rho) = -\frac{4(1 - r_1 - \rho)F_2(\rho)F_3(\rho)}{(1-\rho)^5\rho^5((1 - r_1)^2 + r_1^3(2 - r_1))^2},$$

where

$$\begin{aligned} F_2(\rho) &= (1-\rho)^2((1 - r_1)(1 - r_1^3) + \rho(1 - r_1^3) + \rho^2(1 + r_1 + r_1^2)) \\ &\quad + \rho^3(1 - \rho + (2 - \rho)r_1)r_1(3 - 3r_1 + r_1^2), \\ F_3(\rho) &= (1 - r_1)^2((1 - 3\rho + 3\rho^2)(1 - r_1^3) - \rho^3) + \rho^3 r_1^3(4 - 5r_1 + 2r_1^2). \end{aligned}$$

It is immediate that $F_2(\rho) > 0$ for every $0 < \rho < 1 - r_1$. We also have $F_3(\rho) > 0$ for every $\rho \in (0, 1-r_1)$, since $F_4(\rho) := (1-3\rho+3\rho^2)(1-r_1^3)-\rho^3$ satisfies $F_4'(\rho) = -3((\rho-1+r_1^3)^2+r_1^3(1-r_1^3)) < 0$ and $F_4(1 - r_1) = 3(1 - r_1)r_1^4 > 0$. Therefore, F_1 is strictly decreasing on the interval $(0, 1 - r_1)$, $F_1(0^+) = +\infty$ and $F_1(1 - r_1) = -2\epsilon < 0$.

We still need to check that $F_1(1 - r_1 - \epsilon^{1/2}) > 0$. Expanding it in $\epsilon^{1/2}$ near $\epsilon = 0$, we obtain

$$F_1(1 - r_1 - \epsilon^{1/2}) = \frac{2f(r_1)\epsilon}{((1 - r_1)^2 + r_1^3(2 - r_1))^2} + O(\epsilon^{3/2}),$$

where $f(r_1) = 5(7-8r_1+6r_1^2)+r_1(2+5r_1^3+3r_1^6)+r_1^2(14+6r_1^2+4r_1^3+r_1^5)(1-r_1) > 0$ for $r_1 \in (0,1)$. Hence $F_1(1-r_1-\epsilon^{1/2}) > 0$ for every $\epsilon > 0$ sufficiently small. We conclude that if $\epsilon > 0$ is sufficiently small, then $dH \cdot Y_e > 0$ on $H^{-1}(L_1(\mu)+\epsilon) \cap \{(q,p), q \in \mathcal{D}_\epsilon\}$, and the proof is finished. □

Take (μ,ϵ), $\epsilon > 0$, close to $(\mu_0, 0^+)$, and let Y_2 be the Liouville vector field defined in (5.10), where $b \leq 1/2$. We aim at interpolating Y_2 and Y_e to obtain a Liouville vector field Y_ϵ which is positively transverse to $H^{-1}(L_1(\mu)+\epsilon)$ for every $\epsilon > 0$ sufficiently small.

Consider x coordinates as before, $Vx^T = (\hat{p},\hat{q})^T$, where the symplectic matrix V is given in (5.6) and $(\hat{p},\hat{q})$ denote the re-scaled coordinates with origin at $l_1(\mu)$. The canonical symplectic form ω_0 is given by $dx_1 \wedge dx_3 + dx_2 \wedge dx_4$. Let $c_0 > 0$. Then $dH_2 \cdot Y_2 = c_0 + \lambda_1(1/2-b)(x_1^2+x_3^2) > 0$ on $H_2^{-1}(c_0)$, which implies that Y_2 is everywhere transverse to $H_2^{-1}(c_0)$.

In coordinates x, Y_e becomes

$$\begin{aligned} Y_e &= (\bar{q}_1 + (1-r_1))\partial_{\bar{q}_1} + \bar{q}_2\partial_{\bar{q}_2} = \hat{q}\partial_{\hat{q}} + \epsilon^{-1/2}(1-r_1)\partial_{\hat{q}_1} \\ &= (xQ_0 + \epsilon^{-1/2}(0,d_5,d_6,0))\partial_x^T, \end{aligned}$$

where

$$Q_0 := \begin{pmatrix} 1-d_1 & 0 & 0 & d_3 \\ 0 & 1-d_2 & d_3 & 0 \\ 0 & d_4 & d_1 & 0 \\ d_4 & 0 & 0 & d_2 \end{pmatrix},$$

and

$$\begin{aligned} d_1 &:= 1 + \frac{C_1^2-2}{C_0^2} > 0, & d_2 &:= \frac{C_2^2-2}{C_0^2} > 0, \\ d_3 &:= \frac{(C_2^2-2)C_1\sqrt{\lambda_2}}{C_0^2 C_2\sqrt{\lambda_1}} > 0, & d_4 &:= \frac{(2-C_1^2)C_2\sqrt{\lambda_1}}{C_0^2 C_1\sqrt{\lambda_2}} < 0, \\ d_5 &:= -\frac{(2(a-1)+C_1^2)(1-r_1)}{C_0 C_2\sqrt{\lambda_2}} < 0, & d_6 &:= \frac{(2(a-1)+C_2^2)(1-r_1)}{C_0 C_1\sqrt{\lambda_1}} > 0. \end{aligned}$$

Here, C_1, C_2, C_3 are as in (5.5), and $a = a(r_1)$, see (5.2).

Since both Y_e and Y_2 are Liouville vector fields, there exists a function G defined near $x = 0$ such that $\iota_{Y_e - Y_2}\omega = dG$. In particular, $Y_2 - Y_e = X_G$ and

$$\nabla G = (Y_2 - Y_e)J_4 = (x(\Lambda - Q_0) - \epsilon^{-1/2}(0,d_5,d_6,0))J_4\partial_x^T,$$

where $\Lambda := \text{diag}(1-b, 1/2, b, 1/2)$. Hence, we may choose $G(x) = \frac{1}{2}xQ_G x^T + \epsilon^{-1/2}(-d_6x_1 + d_5x_4)$, where

$$Q_G = \begin{pmatrix} 0 & -d_3 & b-d_1 & 0 \\ -d_3 & 0 & 0 & \frac{1}{2}-d_2 \\ b-d_1 & 0 & 0 & d_4 \\ 0 & \frac{1}{2}-d_2 & d_4 & 0 \end{pmatrix}. \tag{5.11}$$

Let $\epsilon > 0$ be small. By Proposition 5.6, we know that $dH \cdot Y_e > 0$ for any $\epsilon^{1/2}x = (\bar{p},\bar{q})(V^T)^{-1} = ((p,q) - l_1(\mu)) \cdot (V^T)^{-1}$, where $(p,q) \in H^{-1}(L_1(\mu)+\epsilon)$ with $|q+\mu| < \hat{l}_1(\mu) + \mu - \epsilon^{1/2}$. We want to interpolate Y_e and Y_2 in order to obtain a Liouville vector field Y_ϵ which is positively transverse to $H^{-1}(L_1(\mu)+\epsilon)$. We shall construct this interpolation in coordinates $x = (x_1,x_2,x_3,x_4)$.

Denote $\mathcal{M}_{\mu,E}^{e\#m}$ shortly as $\mathcal{M}_\epsilon^{e\#m}$, where $\epsilon = E - L_1(\mu) > 0$ is small. For every $\epsilon > 0$, denote by $\mathcal{N}_\epsilon$ the neck region consisting of points $(p,q) \in \mathcal{M}_\epsilon^{e\#m}$ satisfying $|q+\mu| \geq \mu + \hat{l}_1(\mu) - \epsilon^{1/2}$ and $|q-1+\mu| \geq 1-\mu-\hat{l}_1(\mu) - \epsilon^{1/2}$.

Proposition 5.7. *There exist $0 < \delta < N$ so that for every $\epsilon > 0$ sufficiently small, the following statements hold:*

(i) $\mathcal{N}_\epsilon \subset \{(p,q) : |(p,q) - l_1(\mu)| < 10\epsilon^{1/2}\}$. *Moreover, if $(p,q) \in \mathcal{N}_\epsilon$, then $|x_3| < N$.*

(ii) *There exists a Liouville vector field Y_ϵ on a neighborhood of $\mathcal{M}_\epsilon^{e\#m}$ which is transverse to $\mathcal{M}_\epsilon^{e\#m}$, coincides with Y_e on $-2N < x_3 < -N$, with Y_m on $N < x_3 < 2N$ and with Y_2 on $-\delta < x_3 < \delta$.*

Proof. Recall that $l_1(\mu) = (-i\hat{l}_1(\mu), \hat{l}_1(\mu)) \in \mathbb{C}^2$. Near $l_1(\mu)$, consider the re-scaled coordinates $(p,q) = \epsilon^{1/2}(\hat{p},\hat{q}) + l_1(\mu)$. The re-scaled Hamiltonian $H_\epsilon := \epsilon^{-1}(H(\epsilon^{1/2}(\hat{p},\hat{q}) + l_1(\mu)) - L_1(\mu))$ admits the potential $U_\epsilon := \epsilon^{-1}(U(\epsilon^{1/2}\hat{q} + \hat{l}_1(\mu)) - L_1(\mu))$. As $\epsilon \to 0^+$, U_ϵ converges uniformly in C^∞ to $\frac{1}{2}\nabla^2 U(\hat{l}_1(\mu))(\hat{q},\hat{q}) = \frac{1}{2}((2a-1)\hat{q}_2^2 - (4a+1)\hat{q}_1^2)$ on $|\hat{q}| < C$ for any fixed $C > 0$. Let $(p,q) \in \mathcal{N}_\epsilon \subset \mathcal{M}_\epsilon^{e\#m}$. Then $U_\epsilon(\hat{p},\hat{q}) \leq 1$, $(\epsilon^{1/2}\hat{q}_1 + \hat{l}_1(\mu) + \mu)^2 + (\epsilon^{1/2}\hat{q}_2)^2 \geq (\mu + \hat{l}_1(\mu) - \epsilon^{1/2})^2$ and $(\epsilon^{1/2}\hat{q}_1 + \hat{l}_1(\mu) + \mu - 1)^2 + (\epsilon^{1/2}\hat{q}_2)^2 \geq (1 - \mu - \hat{l}_1(\mu) - \epsilon^{1/2})^2$. Assuming that $|\hat{q}| < C$ for a large constant C, and $(p,q) \in \mathcal{N}_\epsilon$, we use these estimates to obtain

$$(2a-1)\hat{q}_2^2 - (4a+1)\hat{q}_1^2 + O(\epsilon^{1/2}) \leq 2$$

and

$$-1 - \frac{\hat{q}_1^2 + \hat{q}_2^2 - 1}{2(\mu + \hat{l}_1(\mu))}\epsilon^{1/2} \leq \hat{q}_1 \leq 1 + \frac{\hat{q}_1^2 + \hat{q}_2^2 - 1}{2(1 - \mu - \hat{l}_1(\mu))}\epsilon^{1/2},$$

where $2 < a = a(\mu) \leq 4$ is given in (5.2). Combining the inequalities above, we obtain

$$\begin{aligned}&(2a-1)(\hat{q}_1^2 + \hat{q}_2^2) \leq 2 + 6a\hat{q}_1^2 + O(\epsilon^{1/2}) \\ &\leq 2 + 6a \max\left\{\left(1 + \frac{\epsilon^{1/2}(\hat{q}_1^2 + \hat{q}_2^2 - 1)}{2(\mu + \hat{l}_1(\mu))}\right)^2, \left(1 + \frac{\epsilon^{1/2}(\hat{q}_1^2 + \hat{q}_2^2 - 1)}{2(1 - \mu - \hat{l}_1(\mu))}\right)^2\right\} + O(\epsilon^{1/2}).\end{aligned}$$

Taking $\epsilon \to 0$, we see that $\hat{q}_1^2 + \hat{q}_2^2 \leq (2+6a)/(2a-1) < 5$. Therefore, for any $\epsilon > 0$ sufficiently small, $|q - \hat{l}_1(\mu)| < \sqrt{5}\epsilon^{1/2}$ for every (p,q) in the neck region and we take $C = \sqrt{5}$. Since $\mathcal{N}_\epsilon$ contains only one connected component, we conclude that $\mathcal{N}_\epsilon \subset \{(p,q), |q - \hat{l}_1(\mu)| < \sqrt{5}\epsilon^{1/2}\}$ for every $\epsilon > 0$ sufficiently small.

Since $\dot{q} = p + iq$, we estimate

$$\begin{aligned}|p + i\hat{l}_1(\mu)|^2 = |\dot{q} - i(q - \hat{l}_1(\mu))|^2 &\leq 2(L_1(\mu) + \epsilon - U(q))) + |q - \hat{l}_1(\mu)|^2 \\ &\leq 2\epsilon - (2a-1)q_2^2 + (4a+1)(q_1 - \hat{l}_1(\mu))^2 + O(\epsilon^{3/2}) + 5\epsilon \\ &\leq (20a + 12)\epsilon < 92\epsilon.\end{aligned}$$

The estimates above imply that if $(p,q) \in \mathcal{N}_\epsilon$, then $|(p,q) - l_1(\mu)| < 10\epsilon^{1/2}$. Since $\epsilon^{1/2}x^T = V^{-1}((p,q) - l_1(\mu))^T$ is an affine map and V only depends on μ, we may take $N := 10\alpha_1$, where α_1 is the largest absolute value of an eigenvalue of V^{-1}. Therefore, if $(p,q) \in \mathcal{N}_\epsilon$, then $|x_3| < N$. This proves (i). In the proof of (ii) below, we may take N even larger if necessary.

Now we construct the interpolation between Y_e, Y_2, and Y_m. We assume that $b = 1/2$, that is Y_2 is a radial vector field both in coordinates $(\hat{p},\hat{q})$ and x. Our interpolation is constructed in coordinates x and supported in $|x_3| < N$. Choose $N > 0$ as above. It follows from (i) and Proposition 5.6 that for every $\epsilon > 0$ sufficiently small, Y_e is transverse to $\mathcal{M}_\epsilon^{e\#m} \setminus \mathcal{N}_\epsilon$. Moreover, recall that Y_2 is globally transverse to $H_2^{-1}(1)$ where $H_2 = \frac{\lambda_1}{2}(x_1^2 - x_3^2) + \frac{\lambda_2}{2}(x_2^2 + x_4^2)$. Also, Y_2 is invariant under the re-scaling of x. Hence, for every $\epsilon > 0$ sufficiently small, Y_2 is transverse to $\mathcal{M}_\epsilon^{e\#m} \cap \{|x| < N\}$.

Let $\beta : \mathbb{R} \to [0,1]$ be a smooth even function, non-increasing on $(-\infty, 0]$, $\beta(x_3) = 1$ near $x_3 = -N$ and $\beta(x_3) = 0$ near $x_3 = 0$. Later, we will impose additional conditions on β. Recall that $G(x) = \frac{1}{2}xQ_Gx^T + \epsilon^{-1/2}(-d_6x_1 + d_5x_4)$, where Q_G is given in (5.11). Let

$$Y_\epsilon(x) := Y_2(x) - X_{\beta(x_3)G(x)}.$$

Since $H(p,q) = L_1(\mu) + \epsilon H_2(x) + \epsilon^{3/2}R(x)$ near $l_1(\mu)$, where $R(x)$ contains higher order terms, we have

$$\begin{aligned}dH \cdot Y_\epsilon &= dH \cdot Y_2 + d(\beta G) \cdot X_H = dH \cdot (Y_2 - \beta X_G) + Gd\beta \cdot X_H \\ &= \beta dH \cdot Y_e + (1-\beta)dH \cdot Y_2 + Gd\beta \cdot X_H.\end{aligned} \tag{5.12}$$

In the last identity we used that $Y_2 - X_G = Y_e$. We compute the main terms above

$$\begin{aligned} dH\cdot Y_e &= \epsilon\big(\lambda_1(1-d_1)x_1^2+\lambda_2(1-d_2)x_2^2-\lambda_1 d_1 x_3^2+\lambda_2 d_2 x_4^2 \\ &\quad +(\lambda_1 d_4+\lambda_2 d_3)x_1x_4+(\lambda_2 d_4-\lambda_1 d_3)x_2x_3\big)+\epsilon^{3/2}dR\cdot Y_e \\ &\quad +\epsilon^{1/2}(\lambda_2 d_5 x_2-\lambda_1 d_6 x_3), \\ dH\cdot Y_2 &= \epsilon dH_2\cdot Y_2+\epsilon^{3/2}dR\cdot Y_2=\epsilon+O(\epsilon^{3/2})+\epsilon^{3/2}dR\cdot Y_2, \\ Gd\beta\cdot X_H &= \beta'(x_3)\left(xQ_Gx^T/2+\epsilon^{-1/2}(-d_6x_1+d_5x_4)\right)\left(\epsilon\lambda_1x_1+\epsilon^{3/2}dx_3\cdot X_R\right), \end{aligned} \tag{5.13}$$

Denote by $\mathcal{L}_\epsilon\subset\mathcal{N}_\epsilon$ the neck-region of $\mathcal{M}_\epsilon^{e\#m}$ contained in $-N\le x_3\le N$. Then there exists $R_0>0$ so that $|R(x)|<R_0$ for every $x\in\mathcal{L}_\epsilon$ and every $\epsilon>0$ small. Let $x\in\mathcal{L}_\epsilon\cap\{x_3=-c\}$, where $c\in[0,N]$. Then

$$\lambda_1x_1^2+\lambda_2(x_2^2+x_4^2)=(2+\lambda_1c^2)-2\epsilon^{1/2}R(x)<(3+\lambda_1c^2), \tag{5.14}$$

for every $\epsilon>0$ sufficiently small. Moreover, recall that (5.3) and (5.5) give

$$\begin{aligned} \lambda_1=(a-1+\sqrt{a(9a-4)})^{1/2} &\ge \lambda_2=(1-a+\sqrt{a(9a-4)})^{1/2}, \\ C_1=(2+3a-\sqrt{a(9a-4)})^{1/2} &< C_2=(2+3a+\sqrt{a(9a-4)})^{1/2}, \\ C_0^2 &= 2(a(9a-4))^{1/2}. \end{aligned}$$

Since $C_2^4\lambda_2^2-C_1^4\lambda_1^2=8-24a+44a^2+72a^3>0$, we have

$$0<-d_5=\frac{(2(a-1)+C_1^2)(1-r_1)}{C_0C_2\sqrt{\lambda_2}}\le d_6=\frac{(2(a-1)+C_2^2)(1-r_1)}{C_0C_1\sqrt{\lambda_1}}.$$

Therefore, we estimate the following term of $dH\cdot Y_e$

$$\begin{aligned} &\lambda_2d_5x_2-\lambda_1d_6x_3 \\ &=(1-r_1)\left(\frac{\sqrt{\lambda_1}(2(a-1)+C_2^2)|x_3|}{C_0C_1}-\frac{\sqrt{\lambda_2}(2(a-1)+C_1^2)x_2}{C_0C_2}\right) \\ &\ge\frac{(1-r_1)}{C_0C_1C_2}\left(c\sqrt{\lambda_1}C_2(5a+\sqrt{a(9a-4)})-\sqrt{3+\lambda_1c^2}C_1(5a-\sqrt{a(9a-4)})\right) \\ &\ge\frac{(1-r_1)}{C_0C_1C_2}\left(c\sqrt{\lambda_1}(C_1+C_2)\frac{C_0^2}{2}+5a(c\sqrt{\lambda_1}(C_2-C_1)-\sqrt{3}C_1)\right). \end{aligned}$$

Let $\hat c:=\frac{(3/\lambda_1)^{1/2}C_1}{C_2-C_1}$. Notice that $\hat c$ depends only on μ and if $c=\hat c$, then $c\sqrt{\lambda_1}(C_2-C_1)-\sqrt{3}C_1$ in the last term above vanishes. Choose $N>\hat c$ so that (5.14) still holds for every $\epsilon>0$ sufficiently small. Therefore, for any $c\in[\hat c,N]$, we have

$$\lambda_2d_5x_2-\lambda_1d_6x_3\ge\frac{(1-r_1)c\sqrt{\lambda_1}(C_1+C_2)C_0}{2C_1C_2}=:-\hat v_1x_3>0,$$

where $\hat v_1:=\frac{(1-r_1)\sqrt{\lambda_1}(C_1+C_2)C_0}{2C_1C_2}>0$ only depends on μ. Moreover, it follows from (5.13) that $dH\cdot Y_e-\epsilon^{1/2}(\lambda_2d_5x_2-\lambda_1d_6x_3)=O(\epsilon)$ for every $\epsilon>0$ small. We conclude that

$$\beta dH\cdot Y_e\ge-\epsilon^{1/2}\beta\hat v_1x_3>0\quad\text{on}\quad\mathcal{L}_\epsilon\cap\{-N\le x_3\le-\hat c\}.$$

Our interpolation will be supported in this subset, possibly after taking N even larger.

Since $dH_2\cdot Y_2=H_2=1+O(\epsilon^{1/2})$ on $\mathcal{L}_\epsilon$, we obtain from (5.13) the following estimate

$$\begin{aligned} (1-\beta)dH\cdot Y_2 &= (1-\beta)(\epsilon+O(\epsilon^{3/2})+\epsilon^{3/2}dR\cdot Y_2) \\ &\ge(1-\beta)(\epsilon+O(\epsilon^{3/2})) \\ &\ge(1-\beta)\epsilon/2 \end{aligned}$$

on $\mathcal{L}_\epsilon$, for every $\epsilon>0$ sufficiently small.

Using (5.13) and (5.14), we estimate the dominating term of $Gd\beta \cdot X_H$

$$\begin{aligned} |\beta'(x_3)\lambda_1(-d_6x_1^2 + d_5x_1x_4)| &\leq \lambda_1|\beta'(x_3)| \cdot \left|(d_6 + |d_5|)x_1^2 + |d_5|x_4^2\right| \\ &\leq |\beta'(x_3)| \cdot (d_6 + |d_5|(1 + \lambda_1/\lambda_2))(3 + \lambda_1 c^2) \\ &= |\beta'(x_3)|\hat{v}_2(3 + \lambda_1 x_3^2), \end{aligned}$$

where $\hat{v}_2 := d_6 + |d_5|(1 + \lambda_1/\lambda_2) > 0$ only depends on μ. We conclude from (5.13) that

$$\begin{aligned} Gd\beta \cdot X_H &\geq -\epsilon^{1/2}|\beta'(x_3)|(|\lambda_1(d_6x_1^2 - d_5x_1x_4)| + O(\epsilon^{1/2})) \\ &\geq -\epsilon^{1/2}|\beta'(x_3)|\hat{v}_2(4 + \lambda_1 x_3^2), \end{aligned}$$

on $\mathcal{L}_\epsilon$ for every $\epsilon > 0$ sufficiently small.

Notice that $\hat{v}_1, \hat{v}_2 > 0$ only depend on μ. Combining the estimates above, we see from (5.12) that the inequality $dH \cdot Y_\epsilon > 0$ holds true if

$$|\beta'(x_3)| < -\frac{\hat{v}_1 x_3 \beta(x_3)}{\hat{v}_2(4 + \lambda_1 x_3^2)} + \frac{(1 - \beta(x_3))\epsilon^{1/2}}{2\hat{v}_2(4 + \lambda_1 x_3^2)}, \quad \forall x_3 \in [-N, -\hat{c}]. \tag{5.15}$$

Hence, in order to construct β satisfying (5.15), we first choose

$$\beta(x_3) := \left(\hat{v}_2(4 + \lambda_1 x_3^2)\right)^{\frac{\hat{v}_1}{2\hat{v}_2\lambda_1}} - \left(\hat{v}_2(4 + \lambda_1 \hat{c}^2)\right)^{\frac{\hat{v}_1}{2\hat{v}_2\lambda_1}}, \quad \forall x_3 \in [-\check{c}, -\hat{c}],$$

where $\check{c} > 0$ is the unique point satisfying $\beta(-\check{c}) = 1$. Indeed, it follows from this definition that β satisfies the following properties

$$\beta'(x_3) = \frac{\hat{v}_1 x_3 \beta(x_3)}{\hat{v}_2(4 + \lambda_1 x_3^2)} < 0, \quad \forall x_3 \in [-\check{c}, \hat{c}], \quad \beta(-\check{c}) = 1 \quad \text{and} \quad \beta(-\hat{c}) = 0.$$

In particular, (5.15) holds. Notice that $\check{c}$ is explicitly given by

$$\check{c}^2 = \frac{1}{\hat{v}_2\lambda_1}\left(\left(\hat{c}_2(4 + \hat{c}^2\lambda_1)\right)^{\frac{\hat{v}_1}{2\hat{c}_2\lambda_1}} + 1\right)^{\frac{2\hat{v}_2\lambda_1}{\hat{v}_1}} - \frac{4}{\lambda_1},$$

and does not depend on ϵ. We take N even larger so that $N > \check{c}$ and (5.14) still holds for every $\epsilon > 0$ sufficiently small. Since β decreases from 1 to 0 on $[-\check{c}, -\hat{c}]$, we can change β on small neighborhoods of $x_3 = -\check{c}$ and $x_3 = -\hat{c}$ to obtain a new smooth even function $\beta : \mathbb{R} \to [0, 1]$, still denoted β, so that β has the desired properties, i.e., it is a non-increasing smooth function on $(-\infty, 0]$, coincides with 1 on a neighborhood of $x_3 = -N$ and with 0 on a neighborhood $x_3 = -\hat{c}$. It is important to notice that the change of β near the points $x_3 = -\check{c}$ and $x_3 = -\hat{c}$ can be done in such a way that $|\beta'|$ of the new function is less or equal than $|\beta'|$ of the initial function. Hence, condition (5.15) still holds, and the interpolated Liouville vector field Y_ϵ as constructed above is transverse to $\mathcal{M}_\epsilon^{e\#m}$ for every $\epsilon > 0$ sufficiently small. Finally, we can take $\delta := \hat{c}$. □

Proposition 5.7 implies the following proposition.

Proposition 5.8. *Fix $0 < \mu_0 < 1$. For every (μ, ϵ), $\epsilon > 0$, sufficiently close to $(\mu_0, 0)$, the following assertions hold:*

(i) *There exists a Liouville vector field Y_ϵ defined on a neighborhood of $\mathcal{M}^{e\#m}_{\mu,L_1(\mu)+\epsilon}$, which is everywhere transverse to $\mathcal{M}^{e\#m}_{\mu,L_1(\mu)+\epsilon}$ and induces a contact form $\hat{\alpha}_\epsilon$ so that its Reeb flow is a re-paramerization of the Hamiltonian flow.*

(ii) *There exist two neighborhoods $U_\epsilon \subset \mathcal{U}_\epsilon$ of $l_1(\mu_0)$, such that Y_ϵ coincides with $Y_e = (q_1 + \mu)\partial_{q_1} + q_2\partial_{q_2}$ and $Y_m = (q_1 - 1 + \mu)\partial_{q_1} + q_2\partial_{q_2}$ in $\mathcal{M}^e_{\mu,L_1(\mu)+\epsilon} \setminus \mathcal{U}_\epsilon$ and $\mathcal{M}^m_{\mu,L_1(\mu)+\epsilon} \setminus \mathcal{U}_\epsilon$, respectively, and Y_ϵ coincides with $Y_2 = \frac{1}{2}(x_1\partial_{x_1} + x_2\partial_{x_2} + x_3\partial_{x_3} + x_4\partial_{x_4})$ in $\mathcal{M}_{\mu,L_1(\mu)+\epsilon} \cap U_\epsilon$.*

(iii) *Given any neighborhood $\mathcal{U} \subset \mathbb{R}^4$ of $l_1(\mu)$, α_ϵ is uniformly converging to α_0 on the subset $\mathcal{M}^{e\#m}_{\mu,L_1(\mu)+\epsilon} \setminus \mathcal{U}$ as $\epsilon \to 0$, where α_0 is induced by Y_e and Y_m.*

Finally, Theorem 1.5 directly follows from Propositions 5.8 and 5.3.

6. Proof of Theorem 1.7

For every mass ratio and energy below the first Lagrange value, Birkhoff used the shooting method to prove the existence of a retrograde in the bounded components of the energy surface around each primary.

Theorem 6.1 (Birkhoff [6])**.** *Fix $0 < \mu < 1$. Then for every energy $E < L_1(\mu)$, $\mathcal{M}^e_{\mu,E}$ admits a q_2-symmetric retrograde orbit. A similar statement holds for $\mathcal{M}^m_{\mu,E}$.*

Theorem 6.1 also holds for energies below the critical value $-9/2$ in the Hill's lunar problem, and the retrograde orbit is both symmetric in q_1 and q_2, see Section 8.3 in [24]. We briefly explain the proof of Theorem 6.1 and later show how it can be adapted to energies slightly above the first Lagrange value. We start replacing coordinates (p,q) with $(p+i\mu, q-\mu)$ so that the primaries stay at $0,1 \in \mathbb{C}$. The Hamiltonian becomes

$$H(p,q) = \frac{1}{2}((p_1-q_2)^2 + (p_2+q_1)^2) + U(q),$$

where

$$U(q) = -\frac{\mu}{|q-1|} - \frac{1-\mu}{|q|} - \frac{1}{2}|q-\mu|^2.$$

Let us consider two families of trajectories with initial conditions in $\{(0,p_1,q_1,0), p_1,q_1 < 0\}$ and $\{(0,p_1,q_1,0), p_1,q_1 > 0\}$ at energy E. Using Levi-Civita coordinates to regularize collisions with the primary at 0, one can show that for every $-q_1(0) > 0$ sufficiently small, there exists $t_0 > 0$, depending on $q_1(0)$, such that $q_1(t_0) = 0$, $q_2(t) < 0$ and $\dot{q}_1(t) > 0$ for every $t \in (0,t_0]$. Moreover, $\arg(\dot{q}(t_0)) \to -\pi/4$ as $q_1(0) \to 0^-$. If $q_1(0) > 0$ is sufficiently small, then similar properties hold for the backward flow $t \in [-t_0, 0)$, where $t_0 > 0$ depends on $q_1(0)$. In particular, $q_1(-t_0) = 0$ and $\arg(\dot{q}(-t_0)) \to \pi/4$ as $q_1(0) \to 0^+$. Let $\bar{q}_2 > 0$ be the maximal value of q_2 satisfying $E = U(0,\bar{q}_2)$. The families of solutions above induce real-analytic curves in the interior of the rectangle $Q := [-\pi/2,\pi/2] \times [-\bar{q}_2, 0]$, given by

$$\Gamma_1 := \{(\arg(\dot{q}(t_0)), q_2(t_0)), q_1(0) \in (-\epsilon_0, 0)\}, \quad \Gamma_2 := \{(\arg(\dot{q}(-t_0)), q_2(-t_0)), q_1(0) \in (0,\epsilon_0)\},$$

where $\epsilon_0 > 0$ is sufficiently small.

For every energy E up to slightly above $L_1(\mu)$, there exists the least value $\bar{q}_1 < 0$ of q_1 in the bounded component of the Hill region around $(0,0)$ such that $E = U(\bar{q}_1, 0)$. Moreover, using Levi-Civita coordinates again, one shows that for every $q_1(0) - \bar{q}_1 > 0$ sufficiently small, the trajectory $q(t)$ returns to the q_1-axis without touching the q_2-axis. By continuity, there exists a minimal $\bar{q}_1^* \in (\bar{q}_1, 0)$ such that every trajectory $q(t)$ with energy E and initial conditions $q(0) \in (\bar{q}_1^*, 0) \times \{0\}$, $\dot{q}_2(0) < 0, \dot{q}_1(0) = 0$, satisfies

$$q(t)|_{t\in(0,t_0)} \in \{q_1, q_2 < 0\}, \quad \dot{q}_1(t)|_{t\in(0,t_0]} > 0, \quad q_1(t_0) = 0.$$

If $q_1(0) = \bar{q}_1^*$, then at least one of the following conditions holds

(a) $q(t_0) = (0,0)$.
(b) $\exists\, t_1 \in (0,t_0)$ so that $q_1(t_1) < 0$ and $q_2(t_1) = \dot{q}_2(t_1) = 0$.
(c) $\exists\, t_1 \in (0,t_0)$ so that $q_1(t_1), q_2(t_1) < 0$, $\dot{q}_1(t_1) = 0$.
(d) $\dot{q}_1(t_0) = 0$,

where t_0 depends on $q_1(0)$. Cases (b) and (c) can be ruled out by a monotonicity argument proved by Birkhoff in [6]. In case (d), we necessarily have $\dot{q}_2(t_0) > 0$. Hence, we can extend Γ_1 to a real-analytic curve defined on the interval $(\bar{q}_1^*, 0)$ denoted

$$\Gamma_1 = \{(\arg(\dot{q}(t_0)), q_2(t_0)), q_1(0) \in (\bar{q}_1^*, 0)\} \subset Q \setminus \partial Q.$$

If $q_1(0) \to 0^+$, Γ_1 converges to $(-\pi/4, 0) \in \partial Q$. If $q_1(0) \to \bar{q}_1^*$, then either Γ_1 converges to $(\pi/4 + e, 0)$ for some $0 < e \leq \pi/4$ (case (a)), or to $(\pi/2, q_2(t_0))$ for some $-\bar{q}_2 < q_2(t_0) < 0$ (case (d)). This is all proved in [6].

A similar argument holds for the family Γ_2. More precisely, there exists $\hat{q}_1^* \in (0, l_1(\mu)+\mu)$ such that the backward trajectory $q(t)$ with energy E up to $L_1(\mu)$ and initial conditions $q(0) \in (0, \hat{q}_1^*)$,

$\dot{q}_1(0) = 0, \dot{q}_1(0) > 0$, satisfies

$$q(t)|_{t\in(-t_0,0)} \in \{q_2 < 0 < q_1\}, \quad \dot{q}_1|_{t\in[-t_0,0)} > 0, \quad q_1(-t_0) = 0,$$

where t_0 depends on $q_1(0)$. Moreover, for $q_1(0) = \hat{q}_1^*$ at least one of the following cases happen:

(e) $q(-t_0) = (0, 0)$.
(f) $\exists\, t_1 \in (-t_0, 0)$ so that $q_1(t_1) > 0$ and $q_2(t_1) = \dot{q}_2(t_1) = 0$.

As in (b), case (f) can be ruled out. Hence, we extend Γ_2 to a real-analytic curve defined on the interval $(0, \hat{q}_1^*)$ denoted

$$\Gamma_2 = \{(\arg(\dot{q}(-t_0)), q_2(-t_0)), q_1(0) \in (0, \hat{q}_1^*)\} \subset Q \setminus \partial Q.$$

If $q_1(0) \to 0^+$, then Γ_2 converges to $(\pi/4, 0) \in \partial Q$. If $q_1(0) \to \hat{q}_1^*$, then Γ_2 converges to $(-d-\pi/4, 0)$ for some $0 < d \leq \pi/4$ (case (e)).

By uniqueness of solutions, both Γ_1 and Γ_2 do not admit self-intersections and are properly embedded in $\dot{Q} := Q \setminus \partial Q \equiv \mathbb{R}^2$. Hence, they separate $\dot{Q}$ into twp disjoint open components. Moreover, the starting and ending points of Γ_1 lie in different components of $\dot{Q} \setminus \Gamma_2$ and thus Γ_1 must intersect Γ_2. Each intersection point corresponds to a retrograde orbit. Since both curves are real-analytic, the intersection points are isolated. A crossing point between these two curves is a point where one of the curves, say Γ_1, changes the component of $\dot{Q} \setminus \Gamma_2$. Such an intersection point always exists and is stable in the sense that if μ and E are perturbed, there exists a nearby intersection point for the new parameters. We conclude that at least one crossing point exists between Γ_1 and Γ_2.

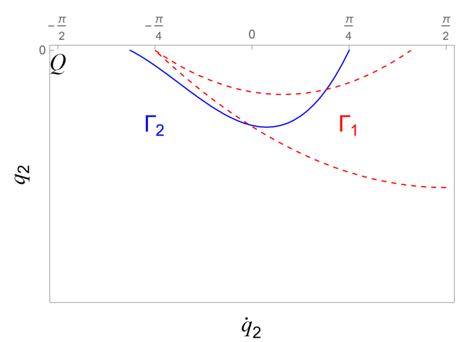

FIGURE 6.1. The curves Γ_1 and Γ_2 in Q

In the following, we shall prove that Birkhoff's shooting method can be used to find retrograde orbits for energies slightly above the first Lagrange value. We start with the following lemma, which states that the endpoint $\hat{q}_1^*$ of the interval $(0, \hat{q}_1^*)$ defined above stays away from $l_1(\mu)$ as the energy increases to slightly above $L_1(\mu)$. In particular, the curves Γ_1 and Γ_2 are well-defined and share the same properties as in the case of lower energies.

Lemma 6.2. *Fix $0 < \mu_0 < 1$. There exists $\delta_0 > 0$ small so that for every (μ, E) sufficiently close to $(\mu_0, L_1(\mu_0))$ the solution in $H^{-1}(E)$ satisfying the initial conditions $q_1(0) = \hat{l}_1(\mu_0) + \mu_0 - \delta_0$, $q_2(0) = 0$, $\dot{q}_1(0) = 0$ and $\dot{q}_2(0) > 0$ satisfies the following conditions backward in time: there exists $t_0 > 0$ so that $q_2(-t_0) = 0$, $q_2(t) < 0 < \dot{q}_1(t)$ for every $t \in (-t_0, 0)$, $0 < q_1(-t_0) < q_1(0)$ and $\dot{q}_2(-t_0) < 0$.*

Proof. It is enough to prove the lemma for $(\mu, E) = (\mu_0, L_1(\mu_0))$ and the statement follows from the transversality properties of the solution at the times where $q_2 = 0$. We look at the solutions of the linearized dynamics near $l_1(\mu_0)$ with initial conditions as in the statement. If δ_0 is taken sufficiently small, we approximate the dynamics in $H^{-1}(L_1(\mu_0))$ by the linearized dynamics at $l_1(\mu_0)$, which is described by the quadratic Hamiltonian $H_2 = H_2(\bar{x})$ given in (5.7), where $\bar{x} := ((p + i\mu_0, q - \mu_0) - l_1(\mu_0))\,(V^T)^{-1}$ and V is the matrix given by (5.6) so that $l_1(\mu)$ corresponds to$\bar{x} = 0$. Hence,

$$q(t) - (\mu_0 + \hat{l}_1(\mu_0), 0) = \left(\frac{2\sqrt{\lambda_2}}{C_2 C_0} \bar{x}_2 + \frac{2\sqrt{\lambda_1}}{C_1 C_0} \bar{x}_3, \frac{C_1}{\sqrt{\lambda_1} C_0} \bar{x}_1 + \frac{C_2}{\sqrt{\lambda_2} C_0} \bar{x}_4 \right)(t).$$

The dynamics of H is given by

$$\begin{aligned} \dot{q}(t) &= p(t) + iq(t) \\ &= \left(\frac{C_2^2 + 2a - 2}{\sqrt{\lambda_1} C_1 C_0} \bar{x}_1 + \frac{C_1^2 + 2a - 2}{\sqrt{\lambda_2} C_2 C_0} \bar{x}_4, \frac{\sqrt{\lambda_2}(C_2^2 - 2)}{C_2 C_0} \bar{x}_2 + \frac{\sqrt{\lambda_1}(C_1^2 - 2)}{C_1 C_0} \bar{x}_3 \right)(t) \\ &+ \left(-\frac{C_1}{\sqrt{\lambda_1} C_0} \bar{x}_1 - \frac{C_2}{\sqrt{\lambda_2} C_0} \bar{x}_4, \frac{2\sqrt{\lambda_2}}{C_2 C_0} \bar{x}_2 + \frac{2\sqrt{\lambda_1}}{C_1 C_0} \bar{x}_3 \right)(t). \end{aligned}$$

Since $q_1(0) = 1 - r_1(\mu_0) - \delta_0$, $\dot{q}_2(0) > 0 = q_2(0) = \dot{q}_1(0)$, we conclude that $\bar{x}_1(0) = \bar{x}_4(0) = 0$ and $-\delta_0 = \frac{2\sqrt{\lambda_2}}{C_2 C_0} \bar{x}_2(0) + \frac{2\sqrt{\lambda_1}}{C_1 C_0} \bar{x}_3(0)$. Recall that $C_2 > C_1 > 0$ for every $\mu \in [0,1]$. Since $H(0, \bar{x}_2, \bar{x}_3, 0) - L_1(\mu_0) = \frac{\lambda_2}{2} \bar{x}_2^2 - \frac{\lambda_1}{2} \bar{x}_3^2 + O(|\bar{x}_2|^3 + |\bar{x}_3|^3) = 0$, we see that

$$\bar{x}_2(0) = \frac{C_0 C_1 C_2}{2\sqrt{\lambda_2}(C_2 - C_1)} \delta_0 + O(\delta_0^2) > 0, \quad \bar{x}_3(0) = -\frac{C_0 C_1 C_2}{2\sqrt{\lambda_1}(C_2 - C_1)} \delta_0 + O(\delta_0^2) < 0,$$

and

$$\dot{q}_2(0) = \frac{\sqrt{\lambda_2} C_2 \bar{x}_2(0) + \sqrt{\lambda_1} C_1 \bar{x}_3(0)}{C_0} = \frac{C_1 C_2}{2} \delta_0 + O(\delta_0^2) > 0,$$

for every $\delta_0 > 0$ sufficiently small.

Consider the re-scaling $\bar{x} = \epsilon^{1/2} x$ variable, we see that $H_\epsilon(x) := \epsilon^{-1}(H(\epsilon^{1/2} x) - L_1(\mu_0))$ converges in C^∞_{loc} to $H_2(x)$. We thus consider a solution $\tilde{x}(t)$ in $H_2^{-1}(0)$ with initial conditions $\tilde{x}(0) = \frac{\delta_0 C_0 C_1 C_2}{2(C_2 - C_1)} (0, \frac{1}{\sqrt{\lambda_2}}, -\frac{1}{\sqrt{\lambda_1}}, 0)$ approximating the solution $x(t)$ in $H_\epsilon^{-1}(0)$ with initial condition $\bar{x}(0)$ mentioned above, where $H_2(\tilde{x}(t)) = \frac{\lambda_1}{2}(\tilde{x}_1^2 - \tilde{x}_3^2) + \frac{\lambda_2}{2}(\tilde{x}_2^2 + \tilde{x}_4^2)(t) = 0$. This is given by

$$\tilde{x}(t) = \frac{\delta_0 C_0 C_1 C_2}{2(C_2 - C_1)} \left(-\frac{\sinh(\lambda_1 t)}{\sqrt{\lambda_1}}, \frac{\cos(\lambda_2 t)}{\sqrt{\lambda_2}}, -\frac{\cosh(\lambda_1 t)}{\sqrt{\lambda_1}}, \frac{\sin(\lambda_2 t)}{\sqrt{\lambda_2}} \right).$$

In q-coordinates up to a re-scaling and a shifting, we write $\tilde{x}(t)$ as $\tilde{q}(t)$, i.e., $V\tilde{x}^T = (\tilde{p}, \tilde{q})^T$. We thus obtain

$$\tilde{q}(t) = \frac{\delta_0 C_1 C_2}{2(C_2 - C_1)} \left(\frac{2\cos(\lambda_2 t)}{C_2} - \frac{2\cosh(\lambda_1 t)}{C_1}, -\frac{C_1 \sinh(\lambda_1 t)}{\lambda_1} + \frac{C_2 \sin(\lambda_2 t)}{\lambda_2} \right),$$

with $\tilde{q}(0) = (-\delta_0, 0)$ and $\dot{\tilde{q}}(0) = (0, \frac{C_1 C_2}{2} \delta_0)$. Except for $t = 0$, $\tilde{q}_2$ admits another zero at $-t_0 \in (-\pi/\lambda_2, 0)$. Therefore, we conclude that for every $\delta_0 > 0$ sufficiently small, the orbit $q(t)$ admits an intersection with q_2-axis at some time $-t_0 \in (-\pi/\lambda_2, 0)$ with $q_1(t_0) < -\delta_0$. To see that $\dot{q}_1|_{t \in (-t_0, 0)} > 0$, we need the following lemma from [6].

Lemma 6.3 (Birkhoff [6, Section 17])**.** *For every* $0 < \mu < 1$*, the inequality* $\partial_{q_1} U > 0$ *holds on* $B_\mu \cap \{0 \leq q_1 < l_1(\mu) + \mu\}$*, where* $B_\mu \subset \mathbb{C} \setminus \{-\mu\}$ *is the projection of* $\mathcal{M}^e_{\mu, L_1(\mu)}$ *to the q-plane.*

From this lemma, we know that $\partial_{q_1} U(q) > 0$ for every $q \in B_{\mu_0} \cap \{0 \leq q_1 < \hat{l}_1(\mu_0) + \mu_0\}$. Then $\frac{d}{dt}(\dot{q}_1 + 2q_2) = -\partial_{q_1} U < 0$ whenever $0 \leq q_1(t) < \hat{l}_1(\mu_0) + \mu_0$, for a solution $(p(t), q(t)) \in \mathcal{M}^e_{\mu_0, L_1(\mu_0)}$. This means that $(\dot{q}_1 + 2q_2)(t)$ is a decreasing function on $t \in [-t_0, 0)$. Since $\dot{q}_1(0) + 2q_2(0) = 0$, we conclude that $(\dot{q}_1 + 2q_2)(t) > 0$ or equivalently $\dot{q}_1(t) > -2q_2(t) > 0$ for every $t \in [-t_0, 0)$. Hence, this lemma holds. □

We are ready to complete the proof of Theorem 1.7. From the arguments above and Lemma 6.2, we know that both Γ_1 and Γ_2 are well-defined real-analytic curves for every (μ, E) sufficiently close to $(\mu_0, L_1(\mu_0))$. Indeed, in the definition of Γ_2, we need the existence of $\hat{q}_1^*$ satisfying the properties above. Lemma 6.2 implies that such $\hat{q}_1^*$ exists for every (μ, E) sufficiently close to $(\mu_0, L_1(\mu_0))$, including energies E greater than $L_1(\mu)$. The curves Γ_1 and Γ_2 do not self-intersect and analytically depend on (μ, E). As mentioned before, they admit at least one crossing point, which is isolated. Such a crossing point varies continuously with (μ, E). For each crossing point, there exists a q_2-symmetric retrograde orbit in $H^{-1}(E)$. If $(\mu, E) \to (\mu_0, L_1(\mu_0))$ the retrograde orbit for (μ, E)

converges in C^∞ to the one of $(\mu_0, L_1(\mu_0))$ since their initial conditions are arbitrarily close to each other. The proof of Theorem 1.7 is now complete.

7. Proof of Theorem 1.9

In this section, we aim to prove Theorem 1.9 by showing that periodic orbits on the regularized energy surface $\hat{H}^{-1}_{\mu,E}(0)$ passing sufficiently close to $S_\pm(\mu_0)$ have high index. Similar index estimates were found in [15, 17, 28] using a different approach. Due to the antipodal symmetry, it is enough to consider $S_+(\mu_0)$. Before proving Theorem 1.9, we need some relevant index estimates.

It will be convenient to consider the Robbin-Salamon index of Lagrangian and symplectic paths. Let $\mathrm{Lag}(\mathbb{R}^{2n},\omega)$ be the collection of all the Lagrangian subspaces of $(\mathbb{R}^{2n},\omega)$. Fix $V, W \in \mathrm{Lag}(\mathbb{R}^{2n},\omega)$ satisfying $\mathbb{R}^{2n} = V \oplus W$. Let $\Lambda : [a,b] \to \mathrm{Lag}(\mathbb{R}^{2n},\omega)$ be a piecewise C^1 path of Lagrangian subspaces. Denote $\Lambda_t := \Lambda(t), \forall t \in [a,b]$. We call $t_0 \in [a,b]$ a crossing of Λ if $\Lambda_{t_0} \cap V \neq \{0\}$ and $\Lambda_{t_0} \pitchfork W$. For each crossing t_0, we define its crossing form as

$$\Gamma(\Lambda, V, t_0)(v) := \frac{d}{dt}\omega(v, v+w_t)\Big|_{t=t_0}, \qquad \forall v \in \Lambda_{t_0} \cap V, .$$

where $w_t \in W$ is uniquely determined by the condition $v + w_t \in \Lambda_t$ for t close to t_0. Notice that Γ does not depend on the choice of W. We call t_0 a regular crossing if $\Gamma(\Lambda, V, t_0)$ is non-degenerate as a quadratic form. In that case, we denote by $\mathrm{Sign}\Gamma(\Lambda, V, t_0)$ the signature of the crossing form at t_0, i.e., the difference between the number of positive and negative eigenvalues. Generically, Λ admits only finitely many regular crossings, and the Robbin-Salamon index of Λ is defined as the half-integer

$$\mu(\Lambda, V) := \frac{1}{2}\mathrm{Sign}\Gamma(\Lambda, V, a) + \sum_{t_0 \in (a,b) \text{ is crossing}} \mathrm{Sign}\Gamma(\Lambda, V, t_0) + \frac{1}{2}\mathrm{Sign}\Gamma(\Lambda, V, b), \tag{7.1}$$

where the first and last terms on the right-hand side only appear if the points a and b are crossings, respectively. Every Lagrangian path is homotopic with fixed ends to a regular path.

Since the diagonal $\Delta = \{(v,v) | v \in \mathbb{R}^{2n}\}$ and, more generally, the graph $\mathrm{Gr}(M) = \{(v, Mv) | v \in \mathbb{R}^{2n}\}$, $M \in \mathrm{Sp}(2n)$, are Lagrangians of $\mathrm{Lag}(\mathbb{R}^{2n} \oplus \mathbb{R}^{2n}, -\omega_0 \oplus \omega_0)$, we define the Robbin-Salamon index of a piecewise C^1 path $\psi : [a,b] \to \mathrm{Sp}(\mathbb{R}^{2n})$ of symplectic matrices by

$$\mu_{\mathrm{RS}}(\psi) := \mu(\mathrm{Gr}(\psi), \Delta) \in \frac{1}{2}\mathbb{Z}.$$

Notice that ψ does not necessarily start from the identity. In particular, $\det(I - \psi(t_0)) = 0$ for each crossing $t_0 \in [a,b]$. The path ψ determines a path of symmetric matrices $S(t) := -J\dot{\psi}(t)\psi^{-1}(t), t \in [a,b]$. Conversely, given a path of symmetric matrices $S(t), t \in [a,b]$, and $\psi_a \in \mathrm{Sp}(2n)$, one can integrate $\dot{\psi} = -JS(t)\psi$ to obtain a path $\psi(t) \in \mathrm{Sp}(2n), t \in [a,b]$ with $\psi(a) = \psi_a$. In that case,

$$\Gamma(t_0) := \Gamma(\mathrm{Gr}(\psi), \Delta, t_0) = S(t_0)|_{\ker(I - \psi(t_0))}$$

is nondegenerate for every regular crossing $t_0 \in [a,b]$. If all crossings are regular, the Robbin-Salamon index of the path ψ is given by

$$\mu_{\mathrm{RS}}(\psi) = \frac{1}{2}\mathrm{Sign}(\Gamma(a)) + \sum_{t_0 \in (a,b) \text{ is crossing}} \mathrm{Sign}(\Gamma(t_0)) + \frac{1}{2}\mathrm{Sign}(\Gamma(b)). \tag{7.2}$$

Among the axiomatic properties of the Robbin-Salamon index, we know that μ_{RS} is invariant under homotopies with fixed end points, is additive under catenation $\mu_{\mathrm{RS}}(\psi) = \mu_{\mathrm{RS}}(\psi|_{[a,c]}) + \mu_{\mathrm{RS}}(\psi|_{[c,b]}), \forall c \in (a,b)$, and satisfies the product property $\mu_{\mathrm{RS}}(\psi \oplus \psi') = \mu_{\mathrm{RS}}(\psi) + \mu_{\mathrm{RS}}(\psi')$. Moreover, if $\psi(a) = I$ and $\psi(b)$ does not have 1 as an eigenvalue, then $\mu_{\mathrm{RS}}(\psi)$ coincides with the Conley-Zehnder index $\mu_{\mathrm{CZ}}(\psi)$ of a nondegenerate path of symplectic matrices. For a general path ψ, with $\psi(a) = I$, the difference between $\mu_{\mathrm{RS}}(\psi)$ and $\mu_{\mathrm{CZ}}(\psi)$ is bounded by a uniform constant depending only on n.

For $2n = 2$, let

$$\psi_1(t) := \begin{pmatrix} \cosh \lambda_1 t & \sinh \lambda_1 t \\ \sinh \lambda_1 t & \cosh \lambda_1 t \end{pmatrix} \psi_1(0), \quad t \in [a,b], \tag{7.3}$$

where $\lambda_1 > 0$ and $\psi_1(0) = (a_{ij})_{2\times 2} \in \mathrm{Sp}(2)$. Then ψ_1 has at most two crossings. This follows from the fact that $\mathrm{tr}(\psi_1(t)) = c_1 e^{\lambda_1 t} + c_2 e^{-\lambda_1 t} = 2$ has at most two solutions $t_1, t_2 \in \mathbb{R}$, where $c_1 := \frac{1}{2}(a_{11}+a_{22}+a_{12}+a_{21})$ and $c_2 := \frac{1}{2}(a_{11}+a_{22}-a_{12}-a_{21})$. The existence of two crossings $t_1 \leq t_2$, with multiplicities counted, only occurs if both coefficients c_1 and c_2 are positive. Otherwise, there exists at most 1 crossing, say t_1, which must be simple, i.e. $\dim\ker(I - \psi_1(t_1)) = 1$. Assume that $c_1, c_2 > 0$. Notice that $S_1 := -J\dot{\psi}_1\psi_1^{-1} = \mathrm{diag}(\lambda_1, -\lambda_1)$. Then $t_1 = t_2$ is equivalent to $c_1c_2 = 1$ and $t_1 \neq t_2$ is equivalent $c_1c_2 < 1$. Assume that $t_1 = t_2$. Since $a_{11}a_{22} - a_{12}a_{21} = 1$, we see that $c_1c_2 = 1$ is equivalent to $|a_{11} - a_{22}| = |a_{12} - a_{21}| =: v$. If $v = 0$, then $a_{11} = a_{22} > 0, a_{12} = a_{21}$. In this case, $\psi_1(t_1) = I$ and the crossing form is S_1, which has signature 0. If $v > 0$, then $\psi_1(t_1)$ has eigenvalue 1 with $\ker(\psi_1(t_1)-I) = \mathbb{R}(1,1)$ or $\mathbb{R}(1,-1)$. Then the crossing form $S_1|_{\ker(\psi_1(t_1)-I)} = 0$. If $t_1 \neq t_2$, we can reduce this case to a small neighborhood of $\{c_1c_2 = 1\}$ by continuity of c_1c_2 and the homotopy invariance of μ_{RS}. Then, as a small perturbation of the previous case, the signature of the crossing forms at t_1 and t_2 is 0. Hence, in all cases, we have

$$|\mu_{\mathrm{RS}}(\psi_1)| \leq 1. \tag{7.4}$$

If

$$\psi_2(t) = \begin{pmatrix} \cos\lambda_2 t & -\sin\lambda_2 t \\ \sin\lambda_2 t & \cos\lambda_2 t \end{pmatrix} \psi_2(0), \quad t \in [a,b], \tag{7.5}$$

for some $\lambda_2 > 0$, then there exist precisely two crossing points $t_1, t_2 \in [a, a + 2\pi/\lambda_2)$ for every $a \in \mathbb{R}$, where multiplicities are counted, i.e., if $t_1 \neq t_2$ then $\dim\ker(I - \psi_2(t_i)) = 1, i = 1,2$, and if $t_1 = t_2$, then $\dim\ker(I - \ker\psi_2(t_i)) = 2$. Since $S_2 := -J\dot{\psi}_2\psi_2^{-1} = \mathrm{diag}(\lambda_2, \lambda_2) > 0$, we have $\sum_t \mathrm{Sign}(\Gamma(t)) = 2$, where the sum is over the crossing points $t \in [a, a + 2\pi/\lambda_2)$. Hence

$$\mu_{\mathrm{RS}}(\psi_2) \geq 2\lfloor \lambda_2(b-a)/2\pi \rfloor, \quad \forall a < b. \tag{7.6}$$

Now, for convenience, we denote $\hat{H} = \hat{H}(y,x)$ the regularized Hamiltonian omitting the subscripts μ, E. Consider a smooth path $\psi(t) \in \mathrm{Sp}(2n), t \in [a,b]$, satisfying $\dot{\psi} = JB(t)\psi$, where $B(t) = \nabla^2\hat{H}(\gamma(t))$ is the Hessian of $\hat{H}$ along a solution $\gamma \subset \hat{H}^{-1}(0)$ of $\dot{\gamma}(t) = J\nabla\hat{H}(\gamma(t))$. Since $S_+(\mu)$ is a saddle-center singularity, see section 5.1, there exist symplectic coordinates $\tilde{x} \in \mathbb{R}^4$ near $S_+(\mu)$ so that the Hamiltonian has the form $\hat{H} = \hat{H}_2 + \hat{R}$, where $\hat{H}_2 = \frac{\hat{\lambda}_1}{2}(\hat{x}_1^2 - \hat{x}_3^2) + \frac{\hat{\lambda}_2}{2}(\hat{x}_2^2 + \hat{x}_4^2)$, and $\hat{R}$ vanishes up to second order. If $\gamma(t) \equiv S_+(\mu), \forall t \in \mathbb{R}$, then $B_{S_+(\mu)}(t) \equiv \mathrm{diag}(\hat{\lambda}_1, \hat{\lambda}_2, -\hat{\lambda}_1, \hat{\lambda}_2)$. In particular, the linearized flow over $S_+(\mu)$ decouples $\psi_{S_+(\mu)} = \psi_1 \oplus \psi_2$ on the $(\hat{x}_1, \hat{x}_3)$ and $(\hat{x}_2, \hat{x}_4)$ directions, as in (7.3) and (7.5), respectively. From (7.4), (7.6), and the product property of the Robbin-Salamon index, we obtain

$$\mu_{\mathrm{RS}}(\psi_{S_+(\mu)}|_{[a,b]}) = \mu_{\mathrm{RS}}(\psi_1|_{[a,b]}) + \mu_{\mathrm{RS}}(\psi_2|_{[a,b]}) \geq 2\lfloor \lambda_2(b-a)/2\pi \rfloor - 1, \quad \forall a < b. \tag{7.7}$$

We observe that the restriction of $\nabla^2\hat{H}$ to $\mathbb{R}\{\partial_{p_1}, \partial_{p_2}\}$ is always positive-definite. Using the above Robbin-Salamon index, a uniform estimate of the Conley-Zehnder index can be obtained.

Proposition 7.1. *Let $x(t) = \varphi_t(x(0))$ be a Hamiltonian trajectory of $\hat{H} = \hat{H}(y,x)$. Let $\psi_t := D\varphi_t(x(t))\psi_0$, $\forall t \in [0,T]$ be the linearized path of symplectic matrices starting from $\psi_0 \in \mathrm{Sp}(4)$. Then $\mu_{\mathrm{RS}}(\psi_t|_{[0,T]}) \geq -9$.*

Proof. Recall from (7.2) that $\mu_{\mathrm{RS}}(\psi_t|_{[0,T]})$ is the sum of $\mathrm{Sign}(\Gamma(t))$ at the crossings $t \in [0,T]$. Let t_0 be a crossing. We have $\ker(I - \psi_{t_0}) \cong \mathrm{Gr}(\psi_{t_0}) \cap \Delta$, where $\mathrm{Gr}(\psi_t) := \{(v, \psi_t(v)) | v \in \mathbb{R}^4\}$ and $\Delta := \{(v,v) | v \in \mathbb{R}^4\}$. Let $\mathcal{L}(4)$ denote the collection of Lagrangian subspaces of $(\mathbb{R}^4 \oplus \mathbb{R}^4, -\omega_0 \oplus \omega_0)$. Then $\mathrm{Gr}(\psi_t), \Delta \in \mathcal{L}(4)$. By definition, we have $\mu_{\mathrm{RS}}(\psi_t|_{[0,T]}) = \mu(\mathrm{Gr}(\psi_t|_{[0,T]}), \Delta)$.

Let L(2) be the collection of Lagrangian subspaces in $(\mathbb{R}^4, \omega_0)$. Consider the Lagrangian subspace $V = \mathbb{R}\{\partial_{y_1}, \partial_{y_2}\} \in \mathrm{L}(2)$. By flowing V, we obtain a Lagrangian path $\psi_t\psi_0^{-1}V|_{[0,T]}$ in L(2). Since $\dot{\psi}_t = J\nabla^2\hat{H}(x(t))\psi_t$ for every t and $\nabla^2\hat{H}|_V = I$, we compute for every crossing $t_0 \in [0,T]$

$$\begin{aligned}\Gamma(\psi_t\psi_0^{-1}V|_{[0,T]}, V, t_0)(v) &= \frac{d}{dt}\omega_0(v, \psi_t\psi_{t_0}^{-1}v)\big|_{t=t_0} = (J_4 v, \dot{\psi}_{t_0}\psi_{t_0}^{-1}v) \\ &= (\nabla^2\hat{H}(x(t_0))v, v) = \|v\|^2 > 0,\end{aligned}$$

for every nonzero $v \in (\psi_{t_0}\psi_0^{-1}V) \cap V$. Since $t=0$ is a boundary crossing with multiplicity 2, we conclude that

$$\mu(\psi_t\psi_0^{-1}V|_{[0,T]}, V) = \mu(\Lambda_t|_{[0,T]}, V \oplus V) \geq 1, \quad \Lambda_t := \mathrm{Gr}(\psi_t\psi_0^{-1}) \in \mathcal{L}(4).$$

Let $s(V \oplus V, \Delta; \Lambda_0, \Lambda_T) := \mu(\Lambda_t|_{[0,T]}, \Delta) - \mu(\Lambda_t|_{[0,T]}, V \oplus V)$ be the Hormander index, where $\Lambda_0 = \Delta$ and $\Lambda_T = \mathrm{Gr}(\psi_T\psi_0^{-1})$, see Theorem 3.5 in [61]. We thus obtain

$$\begin{aligned}\mu_{\mathrm{RS}}(\psi_t\psi_0^{-1}|_{[0,T]}) = \mu(\Lambda_t|_{[0,T]}, \Delta) &= \mu(\Lambda_t|_{[0,T]}, V \oplus V) + s(V \oplus V, \Delta; \Lambda_0, \Lambda_T)\\ &\geq 1 + s(V \oplus V, \Delta; \Delta, \Lambda_T) \geq -5.\end{aligned}$$

The last inequality uses Theorem 3.5-(3) in [61], since $\{V \oplus V, \Delta, \Lambda_T\}$ can be written as the graph of symmetric metrics at the same time, under a suitable change of coordinates.

Choose a path β_1 from ψ_0 to I_4 and let $\beta_2 := \psi_T\psi_0^{-1}\beta_1$. From Appendix A, one can choose β_1 so that $-J\dot{\beta}_1\beta_1^{-1}$ is non-positive-definite and satisfies

$$-4 \leq \mu_{\mathrm{RS}}(\beta_1) \leq 0.$$

In particular, $-J\dot{\beta}_2\beta_2^{-1}$ is also non-positive-definite, then we have $\mu_{\mathrm{RS}}(\beta_2) \leq 0$. Notice that β_2 is a path from ψ_T to $\psi_T\psi_0^{-1}$, and then $\psi_t\psi_0^{-1}$ is a path from I to $\psi_T\psi_0^{-1}$. Notice that the path ψ_t is homotopic to the catenation $\bar{\beta}_2 * \psi_t\psi_0^{-1} * \beta_1$, starting from β_1, where the homotopy is given by $s \mapsto \psi_t\psi_0^{-1}\beta_1(s)$. We conclude from the homotopy invariance and the catenation property that $\mu_{\mathrm{RS}}(\psi_t|_{[0,T]}) = \mu_{\mathrm{RS}}(\beta_1) + \mu_{\mathrm{RS}}(\psi_t\psi_0^{-1}|_{[0,T]}) - \mu_{\mathrm{RS}}(\beta_2) \geq -9$. □

Now we are ready to prove Theorem 1.9. Let $P' = (\gamma, T) \subset \hat{H}^{-1}(0)$ be a T-periodic orbit which is not a cover of the Lyapunov orbit near $S_\pm(\mu)$, where $|\mu - \mu_0|$ and $E - L_1(\mu_0)$ are small. Due to the antipodal symmetry of $\hat{H}$, we only need to consider $S_+(\mu_0)$. Let $\mathcal{U}_1 \subset \mathbb{R}^2 \times (\mathbb{R} \times \mathbb{R}/2\pi\mathbb{Z})$ be a small open neighborhood of $S_+(\mu_0)$ so that if γ intersects $\mathcal{U}_1$, then for (μ, E) sufficiently close to $(\mu_0, L_1(\mu_0))$, γ lies in $\widetilde{\mathcal{M}}_{\mu,E} := \widetilde{\mathcal{M}}^e_{\mu,E} \cup \widetilde{\mathcal{M}}^m_{\mu,E}$. Since γ is not a cover of the Lyapunov orbit near $S_+(\mu)$, we can take $\mathcal{U}_1$ sufficiently small so that γ necessarily intersects $\widetilde{\mathcal{M}}_{\mu,E} \setminus \mathcal{U}_1$. Denote by $\tilde{\alpha}_{\mu,E}$ the lift of the contact form to $\widetilde{\mathcal{M}}_{\mu,E}$, and recall that $\mathcal{A}(\gamma) = \int_\gamma \tilde{\alpha}_{\mu,E}$. Here, the contact form is also well-defined if $E \leq L_1(\mu)$. Let $\psi(t) \in \mathrm{Sp}(4)$ be the path of symplectic matrices obtained by integrating the linearized flow along γ so that $\psi(0) = I$.

Since $S_+(\mu)$ is an equilibrium point that varies smoothly with μ, we can take $\mathcal{U}_1$ even smaller, if necessary, and find an open neighborhood $\mathcal{U}_2 \subset \mathcal{U}_1$ of $S_+(\mu_0)$ so that for (μ, E) sufficiently close to $(\mu_0, L_1(\mu_0))$, the following holds: if γ intersects $\mathcal{U}_2$, then for an arbitrarily large time interval $[a,b]$, γ lies inside $\mathcal{U}_1$ and thus its period is $T > b - a \gg 0$. Taking $\mathcal{U}_1, \mathcal{U}_2$ even small and (μ, E) even closer to $(\mu_0, L_1(\mu_0))$, if necessary, we can assume furthermore that $B(t) := \nabla^2 H(\gamma(t))$ along $\gamma|_{[a,b]}$ is arbitrarily close to the constant path of symmetric matrices $B_{S_+(\mu)} \equiv \nabla^2 H(S_+(\mu))$. Moreover, it follows from (7.7) that given $N \in \mathbb{N}$, there exists a neighborhood $\mathcal{V}_N \subset \mathcal{U}_2$ of $S_+(\mu)$ so that if $\gamma([a,b]) \subset \mathcal{U}_1$ and $\gamma([a,b]) \cap \mathcal{V}_N \neq \emptyset$, then $\mu_{\mathrm{RS}}(\psi|_{[a,b]}) > N$. By Proposition 7.1, the contribution to the Robbin-Salamon index of γ along the complement of $[a,b]$ in the domain of γ is ≥ -9. This follows from the catenation axiom. Then $\mu_{\mathrm{RS}}(\gamma) = \mu_{\mathrm{RS}}(\gamma|_{[a,b]}) + \mu_{\mathrm{RS}}(\gamma|_{[b,T+a]}) > N - 9$. Since $|\mu_{\mathrm{CZ}}(\gamma) - \mu_{\mathrm{RS}}(\gamma)|$ is uniformly bounded by 2, Theorem 1.9 follows by taking the symmetric neighborhood $\mathcal{U}_N := \mathcal{V}_{N+11}$ of $S_\pm$.

8. Proof of Theorem 1.10

Let $\alpha = \alpha_{\mu,E}, J = J_{\mu,E}$ and $\mathcal{S} = \partial\mathcal{M}^e_{\mu,E} = \partial\mathcal{M}^m_{\mu,E}$ be as in Theorem 1.5 for every (μ, E) sufficiently close to $(\mu_0, L_1(\mu_0))$, with $E > L_1(\mu)$. Let $P_3^e = P^e_{3,\mu,E} \subset \mathcal{M}^e_{\mu,E}$ be the continuous family of retrograde orbits and $\mathcal{D} = \mathcal{D}_{\mu,E}$ be the 2-disks for P_3^e as in Theorem 1.7, for (μ, E) sufficiently close to $(\mu_0, L_1(\mu_0))$. Assume that $\mathcal{M}_{\mu_0, L_1(\mu_0)}$ satisfies the conditions given in Theorem 1.10, i.e., the double cover of $P^e_{3,\mu_0,L_1(\mu_0)}$ has index ≥ 3 and $\mathcal{P}_0' = \emptyset$.

First, we show that for (μ, E) sufficiently close to $(\mu_0, L_1(\mu_0))$, with $E > L_1(\mu)$, the regularized component $\mathcal{M}^{e\#m}_{\mu,E}$ does not contain a contractable periodic orbit $P' \neq P_2$ that is unlinked with P_3^e, has rotation number 1 and action $\leq \mathcal{S}(\mathcal{D}, \alpha)$. Fix $N = 3$ and let $\mathcal{U}_3 \subset \mathbb{R}^2 \times (\mathbb{R} \times \mathbb{R}/2\pi\mathbb{Z})$ be the small neighborhood of the singularities $S_\pm(\mu_0)$ corresponding to $l_1(\mu_0)$, as given in Theorem 1.9, so that for (μ, E) sufficiently close to $(\mu_0, L_1(\mu_0))$ every trajectory in $\mathcal{M}^e_{\mu,E}$ which intersects $\mathcal{U}_3$ has index > 3. By contradiction, assume there exists $(\mu_i, E_i) \to (\mu_0, L_1(\mu_0))$, $E_i > L_1(\mu_i)$, and a sequence of contractable periodic orbits $P_i' \subset \mathcal{M}^e_{\mu_i,E_i} \setminus \mathcal{U}_3$ that are unlinked with P_3^e, have rotation number 1 and action $\leq \mathcal{S}(\mathcal{D}, \alpha)$. Passing to a subsequence, P_i' uniformly converge to a contractible periodic orbit $P_0' \subset \mathcal{M}^e_{\mu_0,L_1(\mu_0)} \setminus \{l_1(\mu)\}$ with rotation number 1 and action $\leq \mathcal{S}(\mathcal{D}_{\mu_0,L_1(\mu_0)}, \alpha_{\mu_0,L_1(\mu_0)})$. This orbit is geometrically distinct from P_3^e since the index of $(P_3^e)^2$ is ≥ 3, or equivalently, its rotation number is > 1. Hence, P_0' is unlinked with P_3^e. This contradicts the hypothesis $\mathcal{P}_0' = \emptyset$ and proves (i).

To prove (ii), we recall that the 2-disks $\mathcal{D} \subset \mathcal{M}^e_{\mu,E} \setminus \mathcal{S}$ for P_3^e have $|d\alpha|$-area uniformly bounded by $\mathcal{S}(\mathcal{D}_{\mu_0,L_1(\mu_0)}, \alpha_{\mu_0,L_1(\mu_0)}) + 1$ for every (μ, E) sufficiently close to $(\mu_0, L_1(\mu_0))$. From (i), we can assume that $\mathcal{M}^e_{\mu,E} \setminus \mathcal{S}$ has no periodic orbit with rotation number 1 and action $\leq \mathcal{S}(\mathcal{D}, \alpha)$ for (μ, E) sufficiently close to $(\mu_0, L_1(\mu_0))$. A direct application of Theorem 1.3 gives the desired $2-3$ foliation of $\mathcal{M}^e_{\mu,E}$ for (μ, E) sufficiently close to $(\mu_0, L_1(\mu_0))$, with $E > L_1(\mu)$, which is adapted to the contact form $\alpha = \alpha_{\mu,E}$ and $J' = J'_{\mu,E}$, where J' coincides with J on a neighborhood of $\mathcal{S}$ and is arbitrarily C^∞-close to J, see Theorem 1.5. This finishes the proof of (ii).

Now we prove (iii). Corollary 1.4 gives at least one homoclinic orbit $\gamma \subset \mathcal{M}^e_{\mu,E}$ to P_2 for every (μ, E) sufficiently close to $(\mu_0, L_1(\mu_0))$, with $E > L_1(\mu)$. Let us show that if the branch W^u of the unstable manifold of P_2 in $\mathcal{M}^e_{\mu,E}$ does not coincide with the branch W^s of the stable manifold of P_2 in $\mathcal{M}^e_{\mu,E}$, then there exist infinitely many transverse homoclinic orbits to P_2, which implies infinitely many periodic orbits and positive topological entropy. The proof of this fact goes back to Conley [10] and was further exploited in [16, 17]. Hence, we only briefly address the argument of those references.

Let $(\theta, r) \in \mathbb{R}/2\pi\mathbb{Z} \times (-\epsilon, \epsilon), \epsilon > 0$ small, be suitable real-analytic coordinates on small annuli $A^s, A^u \subset \mathcal{M}^e_{\mu,E}$ transverse to the local branches $W^s_{\text{loc}}, W^u_{\text{loc}}$ of the stable/unstable manifolds W^s, W^u of P_2, respectively at $r = 0$. The points in A^s whose forward trajectory immediately transit to $\mathcal{M}^m_{\mu,E}$ correspond to $r < 0$, and there exists a well-defined local map from $A^s \cap \{r > 0\}$ to $A^u \cap \{r > 0\}$ given by

$$l(\theta, r) = (\theta + \Delta(r), r), \quad \Delta(r) = g(r) - h(r) \ln r, \quad r > 0,$$

where $g(r)$ and $h(r)$ are real-analytic functions defined near $r = 0$ and $h(0) > 0$, see [17, section 4]. This map describes the infinite twist from A^s to A^u as $r \to 0^+$. We may assume that the homoclinic orbit γ locally intersect A^s and A^u precisely at $(\theta, r) = (0, 0)$. The flow determines a real-analytic map $G(\theta, r)$, defined near $(\theta, r) = (0, 0)$ and satisfying $G(0) = 0$, corresponding to the first hit map from A^u to A^s along γ. The image under G of a small arc in $r = 0$ centered $(0, 0)$ is a real-analytic curve $\hat{\eta}_u$ intersecting $r = 0$ only at $(0, 0)$. This is so since W^s does not coincide with W^u and the flow is real-analytic. Also, we may assume that $\hat{\eta}_u$ contains an open sub-arc η_u contained in $r > 0$ and with endpoint in $(0, 0)$, and there exists a small arc η_s in $r = 0$ with an endpoint in $(0, 0)$ which does not lie in the image of $G|_{\{r<0\}}$, see scenarios (b) and (c) in [17, section 4], or the special homoclinic orbits in [16, section 6]. Now the intersections between $l(\eta_u)$ and $G^{-1}(\eta_s)$ correspond to new homoclinic orbits to P_2. In fact, since G is a diffeomorphism onto the image and $l(\eta_u)$ twists infinitely many times around the circle $r = 0$, there exist infinitely many new homoclinic orbits. From the properties of l, it is then a crucial remark from Conley [10] that such intersections are transverse for $r > 0$ sufficiently small. This implies infinitely many transverse homoclinic orbits to P_2, also implying infinitely many periodic orbits and positive topological entropy.

If the branches $W^s, W^u \subset \mathcal{M}^e_{\mu,E}$ coincide, then their intersections with a plane $\mathcal{D} \in \mathcal{F}$ asymptotic to $(P_3^e)^2$ are formed by finitely many embedded circles $C_i, i = 1, \ldots, m$, which bound disjoint closed disks $B_i \subset \mathcal{D}$. Moreover, the $2-3$ foliation in $\mathcal{M}^e_{\mu,E}$ determines a first return area-preserving map h defined in $\mathcal{D} \setminus \cup_i B_i$. The map h^m behaves like the local map l near each C_i in suitable coordinates. Hence, h^m twists infinitely many times around C_i. Frank's generalization of the Poincaré-Birkhoff Theorem, see [22, 23], implies the existence of infinitely many periodic points of h^m, which correspond to infinitely many periodic orbits in $\mathcal{M}^e_{\mu,E}$, see more details in [16, section 7]. This finishes the proof of (iii).

The same argument holds for $\mathcal{M}^m_{\mu,E}$, and if both regularized subset $\mathcal{M}^e_{\mu_0,L_1(\mu_0)}, \mathcal{M}^m_{\mu_0,L_1(\mu_0)}$ satisfy the conditions given in Theorem 1.10, then the union of both $2-3$ foliations for $\mathcal{M}^e_{\mu,E}$ and $\mathcal{M}^m_{\mu,E}$ gives a $3-2-3$ foliation for $\mathcal{M}^{e\#m}_{\mu,E}$ for every (μ,E) sufficiently close to $(\mu_0, L_1(\mu_0))$, with $E > L_1(\mu)$. The proof of Theorem 1.10 is complete.

9. Proof of Theorem 1.12

In this section, we consider the circular planar restricted three-body problem with mass ratio $\mu = 1/2$. We prove that the regularized subsets $\mathcal{M}^e_{\mu,E}$ and $\mathcal{M}^m_{\mu,E}$ are strictly convex for every $E \leq L_1(\mu) = -2$.

To prove the strict convexity of those subsets of the energy surface, we start with a generalization of the results in [64] by presenting conditions for the Hessian of a non-mechanical Hamiltonian to be definite on the tangent space of the energy surface, see Theorem 9.1. This condition is equivalent to the positivity of the sectional curvatures at the regular points. We assume that the magnetic field and the potential are decoupled, which is the case of $\mu = 1/2$. The criterion formula is further simplified, see Corollary 9.3. We then use a certain monotonicity argument to convert the problem of lower energies into that of checking the positivity of a smooth function on the regularized Hill region at the critical level, see Lemma 9.6. The positivity of the function is then checked at the critical level and a simple local-to-global argument shows that positive curvatures at the regular points imply global convexity.

9.1. **The generalized criterion.** Assume that the Hamiltonian H has the following form

$$H(y,x) = \frac{(y_1 + f_1(x))^2}{2} + \frac{(y_2 + f_2(x))^2}{2} + V(x), \quad \forall y, x \in \mathbb{R}^2, \tag{9.1}$$

where $F = (f_1, f_2)$ is a smooth magnetic field and V is a smooth potential function. Let $y_F := y + F$ and $J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. Then

$$\begin{aligned} \nabla H &= (y_1 + f_1, y_2 + f_2, (y+F)\cdot \partial_{x_1} F + \partial_{x_1} V, (y+F)\cdot \partial_{x_2} F + \partial_{x_2} V) \\ &= (y_F, y_F \nabla F + \nabla V), \end{aligned}$$

and

$$\nabla^2 H = \begin{pmatrix} I_2 & \nabla F \\ \nabla F^T & M \end{pmatrix},$$

where $\nabla F = (\partial_{x_1} F^T, \partial_{x_2} F^T)$ and

$$\begin{aligned} M &= \frac{1}{2}\nabla^2(f_1^2 + f_2^2) + y_1 \nabla^2 f_1 + y_2 \nabla^2 f_2 + \nabla^2 V \\ &= \nabla f_1 \otimes \nabla f_1 + \nabla f_2 \otimes \nabla f_2 + (y_1 + f_1)\nabla^2 f_1 + (y_2 + f_2)\nabla^2 f_2 + \nabla^2 V. \end{aligned}$$

Fix the energy surface $\mathcal{M} := H^{-1}(h)$ and assume that $\mathcal{M}$ has at most a finite number of singularities. Denote by S the singular set of $\mathcal{S}$ and notice that $\mathcal{M} \setminus S$ is connected. We assume that the projection of $\mathcal{M}$ to the x-plane is a topological disk $\overline{\mathcal{H}} \subset \mathbb{R}^2$ whose interior $\mathcal{H}$ is formed by points satisfying $y_F \neq 0$ and the points on the boundary $\partial\mathcal{H}$ satisfy $y_F = 0$. For a regular point $(y,x) \in \mathcal{M}$, with $y_F \neq 0 \Rightarrow V(x) < h$, one can choose a global tangent frame

$$\begin{aligned} X_1 &= (-\partial_{y_2} H, \partial_{y_1} H, 0, 0) = (y_F J^T, 0), \\ X_2 &= (-\partial_{x_1} H, -\partial_{x_2} H, \partial_{y_1} H, \partial_{y_2} H) = (-y_F \nabla F - \nabla V, y_F), \\ X_3 &= (\partial_{x_2} H, -\partial_{x_1} H, \partial_{y_2} H, -\partial_{y_1} H) = ((y_F \nabla F + \nabla V)J, y_F J). \end{aligned}$$

We see that $X_1(y,x) \neq 0$ is tangent to the fiber direction whenever $y_F \neq 0$.

Let $W := \nabla^2 H|_{T\mathcal{M}} = (X_i \nabla^2 H X_j^T)_{3\times 3}$ be the 3×3 matrix representing the tangent Hessian of H along $\mathcal{M}$ in the frame X_1, X_2, X_3.

Theorem 9.1. *The matrix W is positive-definite if and only if the determinant of the following 2×2-matrix*

$$U_W(\theta,x_1,x_2) := r^2(\cos\theta\nabla^2 f_1 + \sin\theta\nabla^2 f_2) + r\nabla^2 V + r^{-1}\nabla V\otimes\nabla V, \tag{9.2}$$

is positive. Here, $r = |y_F| = \sqrt{2(h-V)}$ and θ is the argument of y_F. Moreover, if F and V satisfy the following symmetries

$$\begin{aligned} F(x_1,x_2) &= NF(x_1,-x_2) = -NF(-x_1,x_2),\\ V(x_1,x_2) &= V(x_1,-x_2) = V(-x_1,x_2), \end{aligned} \tag{9.3}$$

with $N = \operatorname{diag}(-1,1)$, then

$$U_W(\theta,x_1,x_2) = NU_W(\pi-\theta,x_1,-x_2)N = NU_W(-\theta,-x_1,x_2)N.$$

Proof. A direct computation shows that the entries of $W = (W_{ij})_{3\times 3}$ are given by

$$\begin{aligned} W_{11} &= |y_F|^2,\\ W_{12} &= y_F J(y_F\nabla F + \nabla V)^T - y_F J\nabla F y_F^T,\\ W_{13} &= -y_F(y_F\nabla F + \nabla V)^T + y_F J\nabla F J y_F^T,\\ W_{22} &= |y_F\nabla F + \nabla V|^2 - 2(y_F\nabla F + \nabla V)\nabla F y_F^T + y_F M y_F^T,\\ W_{23} &= (y_F\nabla F + \nabla V)(J\nabla F + \nabla F J)y_F^T - y_F M J y_F^T,\\ W_{33} &= |y_F\nabla F + \nabla V|^2 - 2(y_F\nabla F + \nabla V)J\nabla F J y_F^T - y_F J M J y_F^T. \end{aligned}$$

The determinant of W then becomes

$$\det W = |y_F|^4(C_1 + |y_F|^2 C_2) := |y_F|^4 C_0,$$

where $y_F = \sqrt{2(h-V)}(\cos\theta,\sin\theta)$ and, denoting $(s,t) := (\cos\theta,\sin\theta)$, we have

$$\begin{aligned} C_1 &= (\nabla V J)M(\nabla V J)^T - (\nabla f_1(\nabla V J)^T)^2 - (\nabla f_2(\nabla V J)^T)^2\\ &= \sqrt{2(h-V)}\big(s\cdot(\nabla V J)\nabla^2 f_1(\nabla V J)^T + t\cdot(\nabla V J)\nabla^2 f_2(\nabla V J)^T\big)\\ &\quad + (\nabla V J)(\nabla f_1\otimes\nabla f_1 + \nabla f_2\otimes\nabla f_2 + \nabla^2 V)(\nabla V J)^T\\ &\quad - (\nabla f_1(\nabla V J)^T)^2 - (\nabla f_2(\nabla V J)^T)^2,\\ C_2 &= \det M - (\nabla f_1 J)M(\nabla f_1 J)^T - (\nabla f_2 J)M(\nabla f_2 J)^T + (\nabla f_1(\nabla f_2 J)^T)^2. \end{aligned}$$

Since for any smooth function $g:\mathbb{R}^2\to\mathbb{R}$, we have

$$(\nabla g J)(\nabla f_1\otimes\nabla f_1 + \nabla f_2\otimes\nabla f_2)(\nabla g J)^T = (\nabla f_1(\nabla g J)^T)^2 + (\nabla f_2(\nabla g J)^T)^2, \tag{9.4}$$

we compute

$$\begin{aligned} C_1 &= \sqrt{2(h-V)}\big(s\cdot(\nabla V J)\nabla^2 f_1(\nabla V J)^T + t\cdot(\nabla V J)\nabla^2 f_2(\nabla V J)^T\big)\\ &\quad + (\nabla V J)\nabla^2 V(\nabla V J)^T. \end{aligned} \tag{9.5}$$

Now we shortly denote $f_{k,ij} = \partial_{x_i,x_j} f_k$, $V_i = \partial_{x_i} V$ and $V_{ij} = \partial_{x_i,x_j} V$. Then we obtain

$$\begin{aligned} \det M &= 2(h-V)\det(s\cdot\nabla^2 f_1 + t\cdot\nabla^2 f_2)\\ &\quad + \sqrt{2(h-V)}\cdot\big((s\cdot f_{1,11} + t\cdot f_{2,11})(V_{22} + |\partial_{x_2} F|^2)\\ &\qquad\qquad - 2(s\cdot f_{1,12} + t\cdot f_{2,12})(V_{12} + \partial_{x_1} F\cdot\partial_{x_2} F)\\ &\qquad\qquad + (s\cdot f_{1,22} + t\cdot f_{2,22})(V_{11} + |\partial_{x_1} F|^2)\big)\\ &\quad + \det(\nabla^2 V + \nabla f_1\otimes\nabla f_1 + \nabla f_2\otimes\nabla f_2), \end{aligned}$$

where

$$\begin{aligned} &\det(\nabla^2 V + \nabla f_1\otimes\nabla f_1 + \nabla f_2\otimes\nabla f_2)\\ &= \det\nabla^2 V + (\nabla f_1 J)\nabla^2 V(\nabla f_1 J)^T + (\nabla f_2 J)\nabla^2 V(\nabla f_2 J)^T + (\nabla f_1(\nabla f_2 J)^T)^2. \end{aligned}$$

Using (9.4), we can also compute

$$
\begin{aligned}
&-(\nabla f_1 J)M(\nabla f_1 J)^T - (\nabla f_2 J)M(\nabla f_2 J)^T + (\nabla f_1(\nabla f_2 J)^T)^2 \\
=& -\sqrt{2(h-V)}\big((\nabla f_1 J)(s\cdot\nabla^2 f_1 + t\cdot\nabla^2 f_2)(\nabla f_1 J)^T \\
&\qquad + (\nabla f_2 J)(s\cdot\nabla^2 f_1 + t\cdot\nabla^2 f_2)(\nabla f_2 J)^T\big) \\
&-(\nabla f_1 J)\nabla^2 V(\nabla f_1 J)^T - (\nabla f_2 J)\nabla^2 V(\nabla f_2 J)^T - (\nabla f_1(\nabla f_2 J)^T)^2.
\end{aligned}
$$

The computation above gives

$$
\begin{aligned}
C_2 =& \, 2(h-V)\det(s\cdot\nabla^2 f_1 + t\cdot\nabla^2 f_2) \\
&+ \sqrt{2(h-V)}\big(s\cdot(f_{1,11}V_{22} + f_{1,22}V_{11} - 2f_{1,12}V_{12}) \\
&\qquad + t\cdot(f_{2,11}V_{22} + f_{2,22}V_{11} - 2f_{2,12}V_{12})\big) \\
&+ \det\nabla^2 V.
\end{aligned}
$$

Therefore, we obtain

$$
\begin{aligned}
C_0 =& \, 4(h-V)^2\det(s\cdot\nabla^2 f_1 + t\cdot\nabla^2 f_2) \\
&+ \sqrt{2(h-V)}\big(2(h-V)\big(s\cdot(f_{1,11}V_{22} + f_{1,22}V_{11} - 2f_{1,12}V_{12}) \\
&\qquad + t\cdot(f_{2,11}V_{22} + f_{2,22}V_{11} - 2f_{2,12}V_{12})\big) \\
&\qquad + s\cdot(\nabla VJ)\nabla^2 f_1(\nabla VJ)^T + t\cdot(\nabla VJ)\nabla^2 f_2(\nabla VJ)^T\big) \\
&+ 2(h-V)\det\nabla^2 V + (\nabla VJ)\nabla^2 V(\nabla VJ)^T \\
:=& \, 4(h-V)^2 A_2 + \sqrt{2(h-V)}(2(h-V)A_{13} + A_{11}) + A_0,
\end{aligned}
\tag{9.6}
$$

where

$$
\begin{aligned}
A_2 :=& \, s^2\cdot(f_{1,11}f_{1,22} - f_{1,12}^2) + t^2\cdot(f_{2,11}f_{2,22} - f_{2,12}^2) \\
&+ st\cdot(f_{1,11}f_{2,22} + f_{2,11}f_{1,22} - 2f_{1,12}f_{2,12}), \\
A_{13} :=& \, s\cdot(f_{1,11}V_{22} + f_{1,22}V_{11} - 2f_{1,12}V_{12}) \\
&+ t\cdot(f_{2,11}V_{22} + f_{2,22}V_{11} - 2f_{2,12}V_{12}), \\
A_{11} :=& \, s\cdot(f_{1,22}V_1^2 + f_{1,11}V_2^2 - 2f_{1,12}V_1V_2) \\
&+ t\cdot(f_{2,22}V_1^2 + f_{2,11}V_2^2 - 2f_{2,12}V_1V_2), \\
A_0 :=& \, 2(h-V)(V_{11}V_{22} - V_{12}^2) + V_{22}V_1^2 + V_{11}V_2^2 - 2V_{12}V_1V_2.
\end{aligned}
$$

Note that C_0 admits the same sign as $\det W$, which only depends on V and $\nabla^2 f_i, i = 1,2$. Denoting $r = \sqrt{2(h-V)}$, one can check that $C_0 = D_1 + D_2$, where

$$
\begin{aligned}
D_1 =& -4(h-V)^2(s\cdot f_{1,12} + t\cdot f_{2,12})^2 - 2((h-V)V_{12}^2 + V_{12}V_1V_2) \\
&- 2\sqrt{h-V}(2(h-V)V_{12} + V_1V_2)(s\cdot f_{1,12} + t\cdot f_{2,12}) \\
=& -\big(2(h-V)(s\cdot f_{1,12} + t\cdot f_{2,12}) + \frac{2(h-V)V_{12} + V_1V_2}{\sqrt{2(h-V)}}\big)^2 + \frac{V_1^2V_2^2}{2(h-V)}.
\end{aligned}
$$

and

$$
\begin{aligned}
D_2 =& \, 4(h-V)^2(s^2\cdot f_{1,11}f_{1,22} + t^2\cdot f_{2,11}f_{2,22} + st\cdot(f_{1,11}f_{2,22} + f_{2,11}f_{1,22})) \\
&+ \sqrt{2(h-V)}\big\{2(h-V)\big(s\cdot(f_{1,11}V_{22} + f_{1,22}V_{11}) + t\cdot(f_{2,11}V_{22} + f_{2,22}V_{11})\big) \\
&\qquad + s\cdot(f_{1,22}V_1^2 + f_{1,11}V_2^2) + t\cdot(f_{2,22}V_1^2 + f_{2,11}V_2^2)\big\} \\
&+ 2(h-V)V_{11}V_{22} + V_{22}V_{11}^2 + V_{11}V_2^2 \\
=& \, \big(2(h-V)(s\cdot f_{1,11} + t\cdot f_{2,11}) + \frac{2(h-V)V_{11} + V_1^2}{\sqrt{2(h-V)}}\big)\cdot \\
&\big(2(h-V)(s\cdot f_{1,22} + t\cdot f_{2,22}) + \frac{2(h-V)V_{22} + V_2^2}{\sqrt{2(h-V)}}\big) - \frac{V_1^2V_2^2}{2(h-V)}.
\end{aligned}
$$

Moreover, we see that

$$C_0 = u_{11}(s,t)u_{22}(s,t) - u_{12}(s,t)^2 = \det U_W.$$

where $u_{ij}(s,t) = r^2(s \cdot f_{1,ij} + t \cdot f_{2,ij}) + rV_{ij} + r^{-1}V_iV_j$.

If (y,x) is such that $x \in \partial\mathcal{H}$, then X_1 vanishes. Hence, we consider the simpler frame

$$X_{1b} = (1,0,0,0), \quad X_{2b} = (0,1,0,0), \quad X_{3b} = (0,0,V_2,-V_1).$$

The tangent Hessian in this frame gives the following 3×3 matrix

$$W_b = (X_{ib}\nabla^2 H X_{jb}^T)_{3\times 3} = \begin{pmatrix} 1 & 0 & \nabla f_1(\nabla VJ)^T \\ 0 & 1 & \nabla f_2(\nabla VJ)^T \\ \nabla f_1(\nabla VJ)^T & \nabla f_1(\nabla VJ)^T & (\nabla VJ)M(\nabla VJ)^T \end{pmatrix}$$

Notice that W_b is positive-definite if and only if $\det W_b > 0$. A direct computation shows that

$$\det W_b = (\nabla VJ)M(\nabla VJ)^T - (\nabla f_1(\nabla VJ)^T)^2 - (\nabla f_2(\nabla VJ)^T)^2 = C_1.$$

Since $2(h-V) = 0$ along $\partial\mathcal{H}$, we further obtain from (9.5)

$$\det W_b = (\nabla VJ)\nabla^2 V(\nabla VJ)^T = V_{22}V_1^2 + V_{11}V_2^2 - 2V_{12}V_1V_2,$$

which is independent of the magnetic field F. Finally, W is positive-definite if

$$C_0|_{\mathcal{M}} = C_1 + |y_F|^2 C_2 = \det U_W|_{\mathcal{M}} > 0, \tag{9.7}$$

which also implies that $C_0|_{\partial\mathcal{H}} = C_1|_{\partial\mathcal{H}} = \det W_b|_{\partial\mathcal{H}} > 0$. Hence, the theorem follows. □

Remark 9.2. *If F and V are decoupled, then $D_1 = 0$ and $C_0 = u_{11}(s,t)u_{22}(s,t) - r^{-2}V_1^2V_2^2$. If F vanishes, then $C_0 = A_0$ and the criterion boils down to the mechanical case considered in* [64].

From (9.2), we observe that $\nabla V \otimes \nabla V$ admits two eigenvalues $\{\lambda_1 = |\nabla V|^2, \lambda_2 = 0\}$, which correspond to the eigenvectors $\xi_1 = \nabla V$ and $\xi_2 = \nabla VJ = (V_2, -V_1)$, respectively. If

$$U_1 := r(s\nabla^2 f_1 + t\nabla^2 f_2) + \nabla^2 V > 0,$$

then U_W is positive definite for $(y,x) \in \mathcal{M}$ with $x \in \mathcal{H}$. Moreover, $\det U_W > 0$. If $\nabla^2 F = 0$, then the strict convexity of V implies $\det U_W > 0$.

If F and V are decoupled, then $U_1 = \operatorname{diag}(c(x_1,x_2), d(x_1,x_2))$, where

$$\begin{aligned} c(x_1,x_2) &:= r(x_1,x_2)(sf_{1,11}(x_1) + tf_{2,11}(x_1)) + V_{11}(x_1), \\ d(x_1,x_2) &:= r(x_1,x_2)(sf_{1,22}(x_2) + tf_{2,22}(x_2)) + V_{22}(x_2). \end{aligned} \tag{9.8}$$

If $c, d > 0$, then U_1 is positive definite and $\det U_W > 0$ naturally holds. In general, we have

$$\det U_W = r^{-2}\big((V_1^2 + r^2c)(V_2^2 + r^2 d) - (V_1V_2)^2\big) = dV_1^2 + cV_2^2 + r^2cd.$$

Then $\det U_W > 0$ is equivalent to $dV_1^2 + cV_2^2 + r^2cd > 0$. In summary, we have the following corollary.

Corollary 9.3. *The following properties hold:*

(i) *If $U_1 := r(s\nabla^2 f_1 + t\nabla^2 f_2) + \nabla^2 V > 0$, then U_W is positive definite on points of $\mathcal{M}$ projecting to $\mathcal{H}$. In particular, if $\nabla^2 F = 0$, then $\nabla^2 V|_{\mathcal{H}} > 0$ implies $\det U_W > 0$ on such points.*

(ii) *If F and V are decoupled, then*

$$\det U_W = dV_1^2 + cV_2^2 + r^2cd.$$

where $c = r(sf_{1,11} + tf_{2,11}) + V_{11}$ and $d = r(sf_{1,22} + tf_{2,22}) + V_{22}$.

9.2. **Elliptic coordinates and strict convexity.** We study the convexity of the critical level for $\mu = 1/2$. From the generalized criterion proved in the previous subsection, we have reduced the local convexity problem to that of checking the positivity of a certain smooth function on the Hill region. In this section we prove that this function is positive for the critical energy $L_1(1/2) = -2$ and also for every lower values of energy.

Recall that the Hamiltonian of the circular restricted three-body problem as

$$H_\mu(p,q) = \frac{1}{2}((p_1 - q_2)^2 + (p_2 + q_1)^2) - \frac{\mu}{|q - (1-\mu)|} - \frac{1-\mu}{|q+\mu|} - \frac{1}{2}|q|^2.$$

Shifting the q_1-variable in a way that the primaries stay at $\pm\frac{1}{2}$, and replacing p_2 by $p_2 - 1/2 + \mu$, we obtain the equivalent Hamiltonian

$$\bar{H}_\mu(p,q) = \frac{1}{2}((p_1 - q_2)^2 + (p_2 + q_1)^2) - \frac{\mu}{|q - \frac{1}{2}|} - \frac{1-\mu}{|q + \frac{1}{2}|} - \frac{1}{2}\Big|\frac{1}{2} - \mu + q\Big|^2.$$

Consider the symplectic transformation given by elliptic coordinates

$$p_1 = a_1 y_1 + b_1 y_2, \quad p_2 = a_2 y_1 + b_2 y_2, \quad q_1 = \frac{1}{2}\cosh x_1 \cos x_2, \quad q_2 = \frac{1}{2}\sinh x_1 \sin x_2,$$

where $a_i, b_i, i = 1, 2$, satisfy

$$\begin{aligned}\begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix} &= 2\begin{pmatrix} \sinh x_1 \cos x_2 & \cosh x_1 \sin x_2 \\ -\cosh x_1 \sin x_2 & \sinh x_1 \cos x_2 \end{pmatrix}^{-1} \\ &= \frac{2}{\cosh^2 x_1 - \cos^2 x_2}\begin{pmatrix} \sinh x_1 \cos x_2 & -\cosh x_1 \sin x_2 \\ \cosh x_1 \sin x_2 & \sinh x_1 \cos x_2 \end{pmatrix}.\end{aligned}$$

The regularized Hamiltonian $\hat{H} = \hat{H}_{\mu,h}$ in coordinates $(y, x) \in \mathbb{R}^2 \times (\mathbb{R} \times \mathbb{R}/2\pi\mathbb{Z})$ becomes

$$\begin{aligned}\hat{H}(y,x) &= \frac{1}{4}(\cosh^2 x_1 - \cos^2 x_2)(\bar{H}_\mu(p(y,x), q(x)) - h) \\ &= \frac{1}{2}\left(\left(y_1 + \frac{\sin 2x_2}{8}\right)^2 + \left(y_2 + \frac{\sinh 2x_1}{8}\right)^2\right) + V(x) + (1-2\mu)\hat{V}(x),\end{aligned}$$

where

$$\begin{aligned}V(x) &= -\frac{h\cosh^2 x_1}{4} - \frac{\cosh x_1}{2} - \frac{\sinh^2(2x_1)}{128} + \frac{h\cos^2 x_2}{4} - \frac{\sin^2(2x_2)}{128}, \\ \hat{V}(x) &= \frac{\cos x_2}{2} - \frac{1}{16}(1/2 - \mu + \cosh x_1 \cos x_2)(\cosh^2 x_1 - \cos^2 x_2).\end{aligned}$$

Notice that $\hat{H}$ smoothly depends on μ and h, and the regularized dynamics of $\bar{H}_\mu^{-1}(h)$ corresponds to the dynamics of $\hat{H}^{-1}(0)$ under a double covering map that identifies $(y, x) \sim -(y, x)$.

From now on we fix $\mu = 1/2$. Then

$$\hat{H}(y,x) = \frac{1}{2}\left((y_1 + f_1(x))^2 + (y_2 + f_2(x))^2\right) + V(x).$$

The magnetic field is denoted $F := (f_1, f_2) = \frac{1}{8}(\sin(2x_2), \sinh(2x_1))$ and the potential function is decoupled

$$V(x) = W_1(x_1) + W_2(x_2),$$

where

$$W_1(x_1) = -\frac{h\cosh^2 x_1}{4} - \frac{\cosh x_1}{2} - \frac{\sinh^2(2x_1)}{128}, \quad W_2(x_2) = \frac{h\cos^2 x_2}{4} - \frac{\sin^2(2x_2)}{128}.$$

For every fixed $h \leq -2$, a simple analysis shows that the critical points of V are given by two nondegenerate minima at $(0,0)$ and $(0,\pi)$ with value $-\frac{1}{2}$, two saddle-points at $(0, \pm\pi/2)$ with value $\hat{v}_0 = -h/4 - 1/2 \geq 0$, two saddles at $(\hat{x}_1, 0)$ and $(\hat{x}_1, \pi)$ with value $\hat{v}_1 > \hat{v}_0 > 0$ and two maxima $(\hat{x}_1, \pm\pi/2)$ with value $\hat{v}_2 > \hat{v}_1 > 0$. Notice that $V_1 = \partial_{x_1} W_1 > 0$ on $(0, \hat{x}_1)$. Hence, for $h = -2$, the Hill region $\mathcal{H} = \mathcal{H}^{-2}$ of the bounded subset of the regularized critical level restricted to $\mathbb{R} \times [-\pi/2, \pi/2]$ is bounded by the graphs of $x_1 = \pm f(x_2)$, where $f : [-\pi/2, \pi/2] \to [0, \bar{x}_1]$ is a smooth function on $[-\pi/2, \pi/2]$, and $\bar{x}_1 = f(0) < \hat{x}_1$ is the maximum of f, see Figure 9.1. The boundary of $\mathcal{H}$ has singularities at $(0, \pm\pi/2)$. This subset corresponds to the component around the primary at $q = 1/2$. Since for $\mu = \frac{1}{2}$ the components of the energy surface around the primaries are symmetric and thus admit the same dynamics, we only consider this component. Now observe that $\partial_h V = -\frac{1}{4}(\cosh^2 x_1 - \cos^2 x_2) > 0$ for every $(x_1, x_2) \in [-\bar{x}_1, \bar{x}_1] \times [-\pi/2, \pi/2] \setminus \{(0,0)\}$, we see that for $h < -2$, the Hill region $\mathcal{H}^h$ is an embedded disk contained in the interior of $\mathcal{H}^{-2}$ and, more generally, $\mathcal{H}^{h_2} \subset \text{int}(\mathcal{H}^{h_1})$ for every $h_2 < h_1 \leq -2$.

Due to the symmetry, we restrict the Hill region $\mathcal{H} = \mathcal{H}^h$ to the first quadrant $x_1, x_2 \geq 0$. We denote this subset of $\mathcal{H}$ by $\mathcal{H}_1 = \mathcal{H}_1^h \subset [0, \bar{x}_1] \times [0, \pi/2]$, where $\bar{x}_1(h) := \max\{x_1 : (x_1, x_2) \in \mathcal{H}\}$ satisfies $V(\bar{x}_1, 0) = 0$. Hence $0 < \bar{x}_1(h_2) \leq \bar{x}_1(h_1), \forall h_2 < h_1 \leq -2$. Notice that $W_1(x_1) \geq 0$, $W_2(x_2) \leq 0$ for every $(x_1, x_2) \in [0, \bar{x}_1] \times [0, \frac{\pi}{2}]$. Also, $V_1 = \partial_{x_1} W_1, V_2 = \partial_{x_2} W_2 > 0$ in the interior $[0, \bar{x}_1] \times [0, \frac{\pi}{2}]$. Let $r = r(x) := \sqrt{-2V(x)}$ be defined for $x \in \mathcal{H}$. Then r decreases with both x_1 and x_2.

Let $(s_1, s_2) := (\cosh x_1, \cos x_2)$. Then we write W_1, W_2 as functions of (s_1, s_2)

$$W_1(s_1) = -\frac{s_1}{32}\left(16 + (-1 + 8h)s_1 + s_1^3\right), \quad W_2(s_2) = \frac{s_2^2}{32}((-1 + 8h) + s_2^2).$$

In coordinates (s_1, s_2), $\mathcal{H}_1$ lies in $[1, \bar{s}_1] \times [0, 1]$, where $\bar{s}_1 := \cosh \bar{x}_1$.

If $h = -2$, then $\mathcal{H}_1$ contains the saddle-point $S_+ := (0, \pi/2)$ corresponding to $l_1(1/2)$. Moreover, $\bar{x}_1 \approx 1.19954$ and $\bar{s}_1 \approx 1.80996 < 1.9$ since $V(s_1 = 1.9, s_2 = 1) = 19379/320000 > 0$.

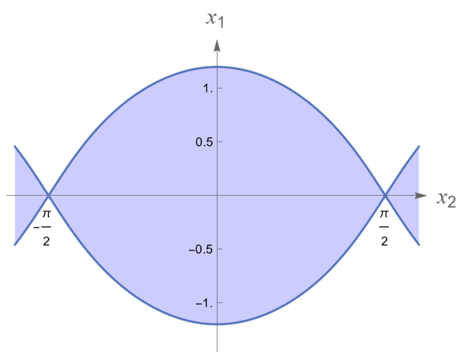

FIGURE 9.1. The Hill region $\mathcal{H} = \mathcal{H}^{-2}$ of the subset $\mathcal{M} = \mathcal{M}^{-2}$ of the regularized critical energy surface.

Let $\mathcal{M} = \mathcal{M}^h$ be the subset of $\hat{H}^{-1}(0)$ projecting to $\mathcal{H}^h$ for every $h \leq -2$. We compute

$$\begin{aligned} r^2 &= \frac{h}{2}(\cosh^2 x_1 - \cos^2 x_2) + \cosh x_1 + \frac{1}{64}(\sinh^2(2x_1) + \sin^2(2x_2)), \\ \nabla^2 f_1 &= \text{diag}\big(0, -\sin x_2 \cos x_2\big), \quad \nabla^2 f_2 = \text{diag}\big(\sinh x_1 \cosh x_1, 0\big), \\ \nabla V &= -\frac{1}{32}\begin{pmatrix} (16 + \cosh x_1 + 16h \cosh x_1 + \cosh(3x_1)) \sinh x_1 \\ (8h + \cos(2x_2)) \sin(2x_2) \end{pmatrix}, \\ \nabla^2 V &= -\frac{1}{16}\text{diag}\big(8\cosh x_1 + 8h\cosh(2x_1) + \cosh(4x_1), 8h\cos(2x_2) + \cos(4x_2)\big). \end{aligned} \tag{9.9}$$

Let $y_F := y + F = re^{i\theta}$ and $(s, t) := (\cos\theta, \sin\theta)$. By Theorem 9.1, the local convexity of $\mathcal{M}$ is equivalent to $\det U_W|_{\mathcal{M}} > 0$. The symmetry condition (9.3) implies that the matrix

$$U_W(\theta, x_1, x_2) = r^2(s\nabla^2 f_1 + t\nabla^2 f_2) + r\nabla^2 V + r^{-1}\nabla V \otimes \nabla V,$$

admits the following symmetries

$$U_W(\theta, x_1, x_2) = U_W(\pi - \theta, x_1, -x_2) \quad \text{and} \quad U_W(\theta, x_1, x_2) = U_W(-\theta, -x_1, x_2). \tag{9.10}$$

Therefore, it suffices to check the local convexity in $\mathcal{M}_1 := \{(\theta, x_1, x_2) \in \mathcal{M} | x \in \mathcal{H}_1\}$.

Our main goal is to prove the following theorem on the local convexity of $\mathcal{M}$.

Theorem 9.4. *The following properties hold:*

(i) *If* $h = -2$*, then* $\det U_W|_{\mathcal{M}\setminus S} > 0$*, where* $S \subset \mathcal{M}$ *is formed by the singularities* $S_\pm$ *corresponding to* $l_1(1/2)$*, and projecting to* $\bar{S}_\pm = (0, \pm\pi/2)$ *in the* x*-plane, respectively.*
(ii) *If* $h < -2$*, then* $\det U_W|_{\mathcal{M}} > 0$*.*

From (9.9), we see that $f_{1,11} = f_{2,22} = 0$ and $-f_{1,22}, f_{2,11} \geq 0$ on $\mathcal{M}_1$. The quantities c, d in (9.8) are reduced to

$$c_t(x) := r \cdot t f_{2,11}(x_1) + V_{11}(x_1), \quad d_s(x) := r \cdot s f_{1,22}(x_2) + V_{22}(x_2).$$

Hence, from Corollary 9.3-(ii), we have

$$\det U_W = d_s(V_1^2 + r^2 c_t) + c_t V_2^2 = c_t\big(V_2^2 + d_s(V_1^2/c_t + r^2)\big). \tag{9.11}$$

We start proving the following lemma.

Lemma 9.5. *For every* $h \leq -2$ *and* $t \in [-1, 1]$*, we have* $c_t \geq c_{-1} = -r f_{2,11} + V_{11} > 0$ *on* $\mathcal{H}_1$*. As a consequence,* $\det U_W|_{\mathcal{M}\setminus S} > 0$ *if and only if* $I := V_2^2 + d_s(V_1^2/c_t + r^2) > 0$ *on* $\mathcal{M}_1 \setminus S$*.*

Proof. We first consider $h = -2$. Since $f_{2,11}(s_1) = s_1\sqrt{s_1^2 - 1}$ is positive on $(1, +\infty)$ and $r = r(s_1, s_2)$ is increasing with s_2, we see for a fixed $s_1 \geq 1$ that $c_t(s_1, s_2) \geq c_{-1}(s_1, s_2) \geq c_{-1}(s_1, 1)$. Hence, it is enough to prove that $c_{-1}(s_1, 1) = \frac{1}{16}(-17 - 8s_1 + 40s_1^2 - 8s_1^4 - 4s_1(s_1^2 - 1)^{1/2}(16 + 16s_1 - 17s_1^2 + s_1^4)^{1/2}) > 0$ for every $s_1 \in [1, \bar{s}_1]$. Indeed, let $v = s_1 - 1 \in [0, \bar{s}_1 - 1] \subset [0, 1]$. We compute

$$\begin{aligned}&(-17 - 8s_1 + 40s_1^2 - 8s_1^4)^2 - 16s_1^2(s_1^2 - 1)(16 + 16s_1 - 17s_1^2 + s_1^4)\\ &= 49 + 48v + 16v^2\big(41(1-v)^4 + 7v(1-v)^5 + 25(1-v)^4v^2 + 29v^6\\ &\quad + 13v(3-4v)^2 + (1-v)v^3(14 - 45v + 44v^2)\big) > 0, \quad \forall v \in [0, 1].\end{aligned}$$

In particular, we see that $c_{-1}(1, s_2) = 7/16 > 0$ for every $s_2 \in [0, 1]$. Therefore, $c_{-1}(s_1, s_2) \geq c_{-1}(s_1, 1) > 0$ for every $(s_1, s_2) \in \mathcal{H}_1 \setminus S$. Notice that c_{-1} is well-defined for $(s_1, s_2) = (1, 0)$ corresponding to $\bar{S}_+$, and is equal to $7/16 > 0$ at this point. Hence, from the symmetries (9.10) and the expression (9.11), the proof is complete for $h = -2$.

For $h < -2$, we change the previous notation to $\mathcal{H}^h, \mathcal{H}_1^h, r_h, V_{i,h}, V_{ii,h} i = 1, 2, c_t^h, d_s^h$. Notice that $\mathcal{H}^h \subset \text{int}(\mathcal{H}^{-2})$ for every $h < -2$. From (9.9), we observe that $r_h^2(x)$ increases with h for every $x \in \mathcal{H}^h$, $\nabla^2 f_1, \nabla^2 f_2$ are independent of h, and $V_{1,h}, V_{2,h}, V_{11,h}$ decrease with h on $\mathcal{H}_1^h$. Hence, $c_t^h(x) \geq c_{-1}^h(x) = -r_h(x) f_{2,11}(x) + V_{11,h}(x) \geq c_{-1}^{-2}(x) > 0$ for every $t \in [-1, 1]$ and $x \in \mathcal{H}_1^h$. Finally, using (9.10) and (9.11), the lemma follows. □

Next we prove Theorem 9.4 using Theorem 9.1, Lemmas 9.5 and 9.6, and a monotonicity argument.

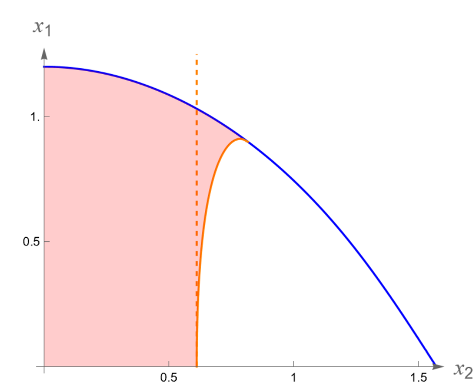

FIGURE 9.2. The boundary curve $\partial\mathcal{H} \cap \mathcal{H}_1$ (blue), the region $\{d_1 > 0\} \cap \mathcal{H}_1$ (red) for $h = -2$, the set $\{d_1 = 0\}$ (orange) and the set $\{D = 0\}$ (orange dashed).

Proof of Theorem 9.4. We keep the previous notation $\mathcal{M}^h, \mathcal{M}_1^h, \mathcal{H}^h, \mathcal{H}_1^h$. Recall that $(s,t) = (\cos\theta, \sin\theta)$ and $I(\theta, x) = V_2^2(x_2) + d_s(x)(V_1^2(x_1)/c_t(x) + r^2(x))$. Here, $(\theta, x) \in \mathcal{M}_1 \setminus S$. By Lemma 9.5, we know that if $d_s(x) > 0$, then $I(\theta, x) > 0$ and thus $\det U_W(\theta, x) > 0$. See the region $\mathcal{H}_1^{-2} \cap \{d_1 > 0\}$ in Figure 9.2. We thus restrict to $\{(\theta, x) \in \mathcal{M}_1^h | d_s(x) \leq 0\}$. In this region, $I \geq V_2^2 + d_s(V_1^2/c_{-1} + r^2)$ since $c_t \geq c_{-1} > 0$ by Lemma 9.5. Moreover, since $f_{1,22} \leq 0$ and $r(x)$ decreases with x_1 for every fixed x_2, we have $\hat{D}(x_2, h) := d_1(0, x_2) \leq d_1(x) \leq d_s(x)$ on $\mathcal{H}_1^h$ for every $s \in [-1,1]$. Notice that $\hat{D}$ depends only on $\cos x_2$ and h. Since $W_0(x) := V_1^2/c_{-1} + r^2 > 0$ on $\mathcal{H}_1^h \setminus S$, it is sufficient to show that $I_0^h(x) := V_2^2 + \hat{D} \cdot W_0 > 0$ on $(\mathcal{H}_1^h \setminus S) \cap \{\hat{D}(x_2) < 0\}$ for every $h \leq -2$. Let $(s_1, s_2) := (\cosh x_1, \cos x_2)$ as before. We compute

$$16\hat{D} = -1 + 8s_2^2 - 8s_2^4 + h(8 - 16s_2^2) - 4s_2(1 - s_2^2)^{\frac{1}{2}}(16 + s_2^2 - s_2^4 + 8h(1 - s_2^2))^{\frac{1}{2}}.$$

Clearly, $I_0^h|_{\mathcal{H}_1^h \setminus S} > 0$ whenever $\hat{D} \geq 0$, since $\hat{D}|_{\{x_2 = 0\}} = (-1 - 8h)/16 > 0$ for every $h \leq -2$.

Now we compute

$$\partial_h W_0(s_1, s_2) = \frac{1}{512c_{-1}^2}\Bigg(\left(\frac{s_1(s_1^2 - s_2^2)}{2r(s_1, s_2)} + (s_1^2 - 1)^{\frac{1}{2}}\right)(s_1^2 - 1)^{\frac{3}{2}}(8 + (-1 + 8h)s_1 + 2s_1^3)^2 + (s_1^2 - 1)\big(s_1(8 + (-1 + 8h)s_1 + 2s_1^3) + 16c_{-1}\big)^2\Bigg) + \frac{1}{2}(1 - s_2^2).$$

We see that $\partial_h W_0 \geq 0$ for every $(s_1, s_2) \in \mathcal{H}_1^h$. Moreover, from (9.9), we see that $r^2(x)$ increases with h for every $x \in \mathcal{H}^h$, $\nabla^2 f_1, \nabla^2 f_2$ are independent of h, and V_1, V_2, V_{11} all decrease with h on $\mathcal{H}_1^h$. Moreover, since $V_1, V_2 \geq 0$ on $\mathcal{H}_1^h$ for every $h \leq -2$, we see that V_1^2, V_2^2 also decrease with h on $\mathcal{H}_1^h$. Hence, for every $(s_1, s_2) \in \Omega_-^h := \{(s_1, s_2) \in \mathcal{H}_1^h : \hat{D}(s_2, h) \leq 0, \partial_h \hat{D}(s_2, h) \leq 0\}$, we have

$$\partial_h I_0^h = \partial_h(V_2^2) + \partial_h \hat{D} \cdot W_0 + \hat{D} \cdot \partial_h W_0 \leq 0.$$

Now we aim to clarify the region $\partial_h \hat{D} \leq 0$ in terms s_2 and h. Write $V = V(s_1, s_2)$. We solve $V(1, s_2) = 0$ for h to obtain

$$\underline{h}(s_2) := -\frac{16 + s_2^2 - s_2^4}{8(1 - s_2^2)},$$

which is decreasing on $[0,1]$ and satisfies $\underline{h}(0) = -2$. Therefore, $\mathcal{H}_1^h \subset \{(s_1, s_2) \in [0, \bar{s}_1] \times [\bar{s}_2, 1]\}$, where $\bar{s}_1(h)$ solves $V(\bar{s}_1, 1) = 0$ and $\bar{s}_2(h)$ is the inverse of $\underline{h}(s_2)$. Let $f(s_2, h) := -32V(1, s_2) = 16 + s_2^2 - s_2^4 + 8h(1 - s_2^2)$. We see that $f > 0$ for every $s_2 \in (\bar{s}_2, 1]$ and $h \leq -2$. Then we compute $\partial_h \hat{D}(s_2, h) = \frac{1}{2} - s_2^2 - s_2(1 - s_2^2)^{\frac{3}{2}} f^{-\frac{1}{2}}$. We solve $\partial_h \hat{D}(s_2, h) = 0$ for h to obtain

$$\bar{h}(s_2) = \frac{16 - 67s_2^2 + 71s_2^4 - 4s_2^6}{8(s_2^2 - 1)(-1 + 2s_2^2)^2}, \quad \forall s_2 \in [0, 1) \setminus \{2^{-\frac{1}{2}}\}.$$

In particular, $\bar{h}(0) = \bar{h}(s_{20}) = -2$, where $s_{20} := \frac{1}{2}(\frac{1}{30}(57 - \sqrt{129}))^{\frac{1}{2}} \approx 0.616726 < 5/8$. Since $\partial_{hh}^2 \hat{D}(s_2, h) = 4s_2(1 - s_2^2)^{\frac{5}{2}} f^{-\frac{3}{2}} > 0$, we know that $\partial_h \hat{D}$ is increasing with $h \in [\underline{h}(s_2), -2]$. Moreover, since $f(s_2, \underline{h}(s_2)) = 0$ and $\partial_h \hat{D}(s_{20}, -2) = 0$, we obtain $\partial_h \hat{D}(s_2, \underline{h}(s_2)) = -\infty$ and $\partial_h \hat{D}(s_2, -2) = \frac{1}{2} - s_2^2 - (1 - s_2^2)^{\frac{3}{2}}(17 - s_2^2)^{-\frac{1}{2}}$, which is positive on $(0, s_{20}]$ and negative on $[s_{20}, 1]$. This implies that for every fixed $s_2 \in (0, s_{20}]$, $\partial_h \hat{D}(s_2, h) < 0$ if and only if $h \in (\underline{h}(s_2), \bar{h}(s_2))$. Moreover, $\partial_h \hat{D}(s_2, h) > 0$ if and only if $h \in (\bar{h}(s_2), -2]$. For every fixed $s_2 \in [s_{20}, 1]$, $\partial_h \hat{D}(s_2, h) < 0$ if and only if $h \in (\underline{h}(s_2), -2]$. See Figure (9.3) for the regions $\{\partial_h \hat{D} < 0\}$ and $\{\partial_h \hat{D} > 0\}$.

Now we consider the region $\Omega_+^h := \{(s_1, s_2) \in \mathcal{H}_1^h | \hat{D} \leq 0, \partial_h \hat{D} \geq 0\}$, where $h \in [\bar{h}(s_2), -2]$ for every $s_2 \in [0, s_{20}]$. We compute

$$\begin{aligned}
\partial_h I_0^h &= \partial_h(V_2^2) + \partial_h \hat{D} \cdot W_0 + \hat{D} \cdot \partial_h W_0 \\
&= \partial_h(V_2^2) + \frac{1}{2}((1 - s_2^2)\hat{W}_0 - s_2^2 W_0) - s_2(1 - s_2^2)^{\frac{3}{2}} f^{-\frac{1}{2}} W_0 \\
&\quad + \frac{\hat{D}}{512c_{-1}^2}\Bigg((\frac{s_1(s_1^2 - s_2^2)}{2r(s_1, s_2)} + (s_1^2 - 1)^{\frac{1}{2}})(s_1^2 - 1)^{\frac{3}{2}}(8 + (-1 + 8h)s_1 + 2s_1^3)^2 \\
&\quad + (s_1^2 - 1)(s_1(8 + (-1 + 8h)s_1 + 2s_1^3) + 16c_{-1})^2\Bigg).
\end{aligned}$$

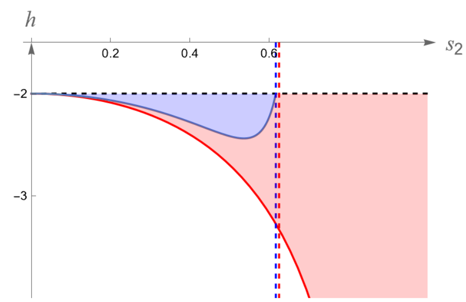

FIGURE 9.3. The regions $\{\partial_h \hat{D} < 0\}$ (red), $\{\partial_h \hat{D} > 0\}$ (blue), in which the blue and red curve denote the graph of $\bar{h}$ on $[0, s_{20}]$ and of $\underline{h}$ on $[0,1]$. The blue and red dashed lines are $s_2 = s_{20}$ and $s_2 = 5/8$, respectively.

where $\hat{W}_0(s_1,s_2,h) := W_0(s_1,s_2,h) + \hat{D}(s_2,h)$. We see that all the terms are non-positive except $\hat{W}_0$ on Ω_+^h. Since $\partial_h W_0 \geq 0$ and $\partial_h \hat{D} \geq 0$ on $\{\partial_h \hat{D} \geq 0\}$, we have $\hat{W}_0(s_1,s_2,h) \leq \hat{W}_0(s_1,s_2,-2)$ for every $\bar{h}(s_2) \leq h \leq -2$. We shall prove that $\hat{W}_0(s_1,s_2,-2) < 0$ on Ω_+^h.

We first compute that

$$\partial_{s_2}\hat{W}_0(s_1,s_2,-2) = \frac{s_2}{8}\Bigg(37 - 18s_2^2 + (20 - 4(1-s_2^2)^{\frac{1}{2}}(17-s_2^2)^{\frac{1}{2}}) + \frac{4s_2^2(9-s_2^2)}{(1-s_2^2)^{\frac{1}{2}}(17-s_2^2)^{\frac{1}{2}}} + \frac{2s_1(-1+s_1^2)^{\frac{3}{2}}}{2^{10}r(s_1,s_2)\cdot c_{-1}^2}(8 - 17s_1 + 2s_1^3)^2(17-2s_2^2)\Bigg) \geq 0,$$

for every $(s_1,s_2) \in \mathcal{H}_1^{-2} \subset [1,\bar{s}_1] \times [0,1]$. Then $\hat{W}_0(s_1,s_2,-2) \leq \hat{W}_0(s_1,s_{20},-2) \leq \hat{W}_0(s_1,\frac{5}{8},-2)$. Now we aim to show that $\hat{W}_0(s_1,5/8,-2) < 0$ for every $s_1 \in [1,\hat{s}_1]$, where $\hat{s}_1 \approx 1.37116$ solves $V(\hat{s}_1,5/8,-2) = 0$. We first compute

$$\begin{aligned} 2^{20}c_{-1}(s_1,5/8)\hat{W}_0(s_1,5/8,-2) &= E_0(s_1) + 4s_1(100\sqrt{41457} - 20361)J(s_1) \\ &+ 4s_1(4418 - 65536s_1 + 69632s_1^2 - 4096s_1^4)(J(s_1) - (5/4 - 8(s_1-5/4)^2)) \\ &+ (100\sqrt{41457} - 20360)(17 + 8s_1 - 40s_1^2 + 8s_1^4). \end{aligned} \tag{9.12}$$

where $J(s_1) := 4(s_1^2-1)^{\frac{1}{2}}r(s_1,5/8) = (s_1^2-1)^{\frac{1}{2}}\left(\frac{26575}{4096} + 16s_1 - 17s_1^2 + s_1^4\right)^{\frac{1}{2}} \geq 0$ for every $s_1 \in [1,\hat{s}_1]$, and

$$\begin{aligned} E_0(s_1) = &-187055 - 163474s_1 + 2863736s_1^2 - 6584384s_1^3 + 6310408s_1^4 - 2469888s_1^5 \\ &+ 98304s_1^6 + 131072s_1^7 - 16384s_1^8. \end{aligned}$$

We see that $20360 < 100\sqrt{41457} \approx 20360.99 < 20361$, $17 + 8s_1 - 40s_1^2 + 8s_1^4 = -7(3-2s_1) - 41(s_1-1) - (s_1-1)(3-2s_1)(-1+10s_1+4s_1^2) < 0$ for every $s_1 \in [1,3/2]$ and $4418 - 65536s_1 + 69632s_1^2 - 4096s_1^4 = 4418 + 4096(s_1-1)s_1(16 - s_1 - s_1^2) > 0$ for every $s_1 \in [1,2]$. Moreover,

$$\begin{aligned} (5/4 - 8(s_1-5/4)^2)^2 - J(s_1)^2 &= (s_1-1)^4\left(\frac{1549675}{4096} - \frac{1048977s_1}{2048} + \frac{713883s_1^2}{4096}\right) \\ &+ (2-s_1)^2\left(\frac{28749}{4096}(6-5s_1)^2(s_1-1)^2 + \frac{2993}{2048}(7-6s_1)^2(s_1-1) + \frac{9}{64}(15-13s_1)^2\right) \\ &+ \frac{44897}{2048}(2-s_1)(s_1-1)^3(5-4s_1)^2 > 0, \quad \forall s_1 \in [1,2]. \end{aligned}$$

We rewrite E_0 as

$$\begin{aligned} E_0(1+v) = &-1024v^3(1-2v)^3(5-v)(13+2v) - 189054v^2(1-2v)^3 \\ &- 185596v^3(1-2v)^2 - 5817v(1-2v)^2 - 29434v^2(1-2v) \\ &- 17665(1-2v)(1-6v)^2 - 162199v(4v-1)^2 < 0, \quad v \in [0,1/2]. \end{aligned}$$

From the estimates above, we see that all the terms in (9.12) are negative in $[1, \hat{s}_1] \subset [1, 3/2]$. We conclude that $\hat{W}_0(s_1, s_2, h) < \hat{W}_0(s_1, \frac{5}{8}, -2) < 0$ on Ω^h_+. Combining both cases $\Omega^h_+ \cup \Omega^h_-$, we obtain $\partial_h I^h_0 \leq 0$ in $\{\hat{D} \leq 0\}$.

Let $h_*(s_2, h) := \min\{h_0 \in [h, -2] : \hat{D}(s_2, h_0) \geq 0\}$, with the convention that $h_*(s_2, h) = -2$ if $\hat{D}(s_2, h_0) < 0$ for every $h_0 \in [h, -2]$. We know that $I^h_0 > 0$ on $\{\hat{D} \geq 0\} \setminus \{\bar{S}_+\}$ and $\partial_h I_0 \leq 0$ on $\{\hat{D} \leq 0\}$. Moreover, by Lemma 9.6 below, $I^{-2}_0 > 0$ on $\mathcal{H}^{-2}_1 \setminus S$. Hence, if $\hat{D}(s_2, h) < 0$, then $I^h_0(s_1, s_2) \geq I^{h_*}_0(s_1, s_2) > 0$. Notice that in this case, we have $\mathcal{H}^h_1 \subset \mathcal{H}^{h_*}_1$. We conclude that $I^h_0(x) > 0$ for every $x \in \mathcal{H}^h_1 \setminus S$. The proof of Theorem 9.4 is complete. □

Lemma 9.6. *Fix $h = -2$, and let $I_0(x) := V_2(x_2)^2 + D(x_2)(V_1(x_1)^2/c_{-1}(x) + r^2(x))$ on $\mathcal{H}_1$, where $D(x_2) := d_1(0, x_2) = r(0, x_2)f_{1,22}(x_2) + V_{22}(x_2)$. Then $I_0(x) > 0$ for every $x \in \mathcal{H}_1 \setminus S$. Moreover, $I_0(x) = O(\cos^4 x_2)$ uniformly in x_1 near S.*

Proof. If $x = (x_1, x_2) \in \mathcal{H}_1 \setminus S$ and $D(x_2) \geq 0$, then $I_0(x) > 0$ naturally holds, see Lemma 9.5. In fact, $V_2(x_2)$ only vanishes at $x_2 = 0$, where $D = 15/16$. Therefore, it is sufficient to consider $x \in \mathcal{H}_1 \cap \{D(x) < 0\}$. See Figure 9.2 for the curve $\{D(x) = 0\}$. Let $(s_1, s_2) = (\cosh x_1, \cos x_2)$ as before and write $D = D(s_2)$, where $s_2 \in [0, 1]$. We compute

$$D(s_2) = \frac{1}{16}(-17 - 8s_2^4 + 40s_2^2 - 4s_2^2(1 - s_2^2)^{1/2}(17 - s_2^2)^{1/2}). \tag{9.13}$$

Hence,

$$D'(s_2) = \frac{s_2\big(8 + (1 - s_2^2)^{1/2}((10 - 4s_2^2)(17 - s_2^2)^{1/2} - (25 - 2s_2^2)(1 - s_2^2)^{1/2})\big)}{2(1 - s_2^2)^{1/2}(17 - s_2^2)^{1/2}}.$$

Since $(10 - 4s_2^2)^2(17 - s_2^2) - (25 - 2s_2^2)^2(1 - s_2^2) = 1075 - 735s_2^2 + 248s_2^4 - 12s_2^6 > 0$ for every $s_2 \in [0, 1]$, we have $D'(s_2) > 0$, and thus $D(s_2)$ is an increasing functions with a unique zero $\hat{s}_2 \in (0, 1)$, since $D(0) = -17/16$ and $D(1) = 15/16$. We also know that $D(s_2) < 0, \forall s_2 \in [0, \hat{s}_2)$. It will be important later to know that $\hat{s}_2 < 0.82$ since $D(0.82) > 0$ as one easily checks. Hence we only need to prove that $I_0(s_1, s_2) > 0$ in $\Omega := \{(s_1, s_2) \in [1, \bar{s}_1] \times [0, \hat{s}_2], V(s_1, s_2) \leq 0\} \subset \mathcal{H}_1$. Notice that Ω corresponds to the region of $\mathcal{H}_1$, where $D \leq 0$.

From $V(s_1, s_2) = \frac{1}{32}(-s_1(16 - 17s_1 + s_1^3) + s_2^4 - 17s_2^2) = 0$, we see that

$$s_1 = 1 + t_0(s_1, s_2)^2 s_2^2, \quad t_0 = \sqrt{\frac{17 - s_2^2}{s_1(16 - s_1 - s_1^2)}},$$

where $t_0(s_1, s_2) \leq \sqrt{17/14} < 6/5$ on $[1, \bar{s}_1] \times [0, 1]$. Notice that $s_1(16 - s_1 - s_1^2)$ is increasing in $s_1 \in [1, \bar{s}_1]$. Consider the change of coordinates $(s_2, c) \mapsto (s_1, s_2)$, where $s_1 = 1 + c^2 s_2^2$, for every $(s_2, c) \in (0, 1] \times [0, \sqrt{17/14}]$. In new coordinates (s_2, c), we have $\mathcal{H}_1 \setminus S \subset (0, 1] \times [0, \sqrt{17/14}]$. From the expression of t_0 above, we see that the curve $V = 0$ in these coordinates, are given by (s_2, t_0), where $t_0 < \sqrt{17/14}$. The constant curve $c(s_2) = \sqrt{17/14}, \forall s_2 \in [0, 1]$, in x-coordinates, is depicted in Figure 9.4. We compute in new coordinates (s_2, c)

$$64c_{-1}(s_2, c)I_0(s_2, c) = s_2^4 E_1(s_2, c), \tag{9.14}$$

where

$$\begin{aligned} E_1(s_2, c) &:= j_1(s_2)c_{-1}(s_2, c) - c^3 D(s_2)W_3(s_2, c), \\ 16c_{-1}(s_2, c) &:= j_0(s_2, c) - 4cs_2^2(1 + c^2 s_2^2)j_2(s_2, c), \\ W_3(s_2, c) &:= j_3(s_2, c)j_2(s_2, c) + cj_4(s_2, c), \end{aligned} \tag{9.15}$$

and

$$\begin{aligned}
j_0(s_2,c) &:= 7+40c^2s_2^2-8c^4s_2^4-32c^6s_2^6-8c^8s_2^8,\\
j_1(s_2) &:= (85-26s_2^2+s_2^4-(17-s_2^2)(1-s_2^2)^{1/2}(17-s_2^2)^{1/2}),\\
j_2(s_2,c) &:= \big((2+c^2s_2^2)(-32s_2^{-2}V(1+c^2s_2^2,s_2)\big)^{1/2}\\
&:= \big((2+c^2s_2^2)(17-14c^2-s_2^2-11c^4s_2^2+4c^6s_2^4+c^8s_2^6)\big)^{1/2},\\
j_3(s_2,c) &:= (-14+3c^2s_2^2+c^4s_2^4)(1+c^2s_2^2)^2,\\
j_4(s_2,c) &:= 70+18c^2s_2^2-105c^4s_2^4-74c^6s_2^6+2c^8s_2^8+8c^{10}s_2^{10}+c^{12}s_2^{12}.
\end{aligned}$$

We have to prove that $E_1(s_2,c)>0$ for every $(s_2,c)\in\mathcal{H}_1\setminus S$ so that $D(s_2)<0$. We start computing $E_1(0,0)=\frac{119}{16}(5-\sqrt{17})>0$. Since E_1 is continuous on $\mathcal{H}_1$, we see that $I_0=O(s_2^4)$ uniformly in c near $(s_2,c)=(0,0)$.

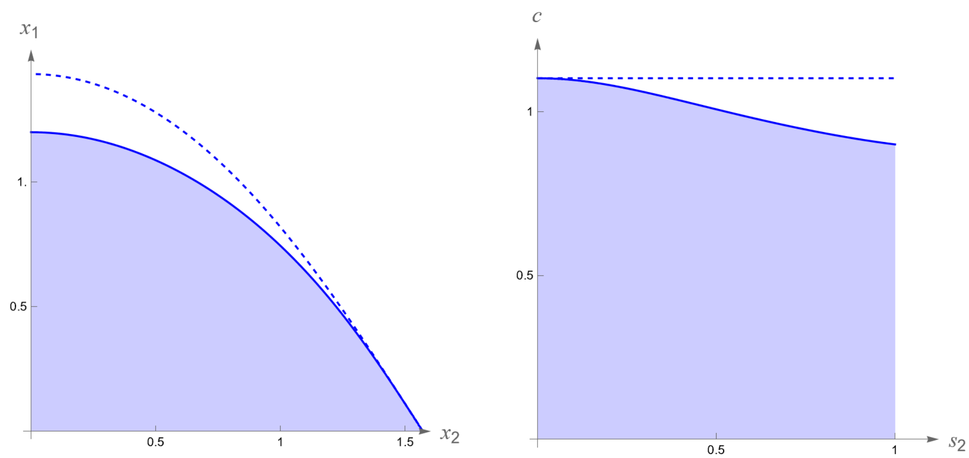

FIGURE 9.4. The blue dashed curve is $s_1=1+\frac{17}{14}s_2^2$ (or $c=\sqrt{17/14}$) and the blue region is $\mathcal{H}_1$ in different coordinates.

Before we continue with the proof of Lemma 9.6, we prove the following lemma.

Lemma 9.7. *The following estimates hold:*

(i) $j_0(s_2,c)>0$ *and* $j_3(s_2,c)<0$ *for every* $(s_2,c)\in\mathcal{H}_1$.
(ii) $j_1(s_2)\geq 85-17^{3/2}>59/4$ *and* $j_4(s_2,c)>0$ *for every* $(s_2,c)\in[0,\hat{s}_2]\times[0,\sqrt{17/14}]$.
(iii) $0\leq|j_2(s_2,c)|<j_{2,0}(s_2,c)$ *for every* $(s_2,c)\in[0,1]\times[0,\sqrt{17/14}]$, *where*

$$j_{2,0}(s_2,c)=\frac{169}{32}-\left(c^2-\frac{1}{4}\right)\left(\frac{5}{2}+\frac{s_2^2}{4}+c^2\right)-\frac{c^4s_2^2}{2}.$$

(iv) $D_-(s_2)\leq 16D(s_2)\leq D_+(s_2)$ *for every* $s_2\in[0,1]$, *where*

$$\begin{aligned}
D_-(s_2) &= -17-8s_2^4+40s_2^2-4s_2^2(33/8-(2s_2^2+s_2^4)),\\
D_+(s_2) &= -17-8s_2^4+40s_2^2-4s_2^2(1-s_2^2)(4+2s_2^2+s_2^4).
\end{aligned}$$

Proof. First, we rewrite j_0 as $j_0(s_2,c)=g_0(1+c^2s_2^2)$, where $g_0(s_1)=-17-8s_1+40s_1^2-8s_1^4$. Since $g_0''(s_1)=80-96s_1^2<0$ for every $s_1\geq 1$, we see that g_0 is a concave function on $[1,+\infty)$. Therefore, $g_0(1)=7$ and $g_0(1.9)=9929/1250>0$ which implies that $g_0(\bar{s}_1)>0$ since $1<\bar{s}_1<1.9$. We conclude that $g_0(s_1)>0$ for every $s_1\in[1,\bar{s}_1]$ since g_0 is concave on the interval. Hence, $j_0>0$ on $\mathcal{H}_1$. Moreover, $j_3<0$ on $\mathcal{H}_1$, since in that region $s_2\in[0,1]$ and $c\in[0,\sqrt{17/14}]$. This proves (i).

To prove (ii), we rewrite and estimate j_1 as follows

$$\begin{aligned} j_1(s_2) &= 85 - 17^{3/2} + s_2^2\left(-26 + s_2^2 + \frac{5780 - 918s_2^2 + 52s_2^4 - s_2^6}{17^{3/2} + (1-s_2^2)^{1/2}(17-s_2^2)^{3/2}}\right) \\ &\geq 85 - 17^{3/2} + s_2^2\left(s_2^2 + \frac{5780 - 26\cdot 2\cdot 17^{3/2} - 918s_2^2 + 52s_2^4 - s_2^6}{2\cdot 17^{3/2}}\right) \\ &\geq 85 - 17^{3/2} + s_2^2\left(s_2^2 + \frac{2000 - 918s_2^2 + 52s_2^4 - s_2^6}{2\cdot 17^{3/2}}\right) \\ &\geq 85 - 17^{3/2} > \frac{59}{4}, \quad \forall s_2 \in [0,1]. \end{aligned}$$

As for j_4, we rewrite it as $j_4(s_2, c) = g_4(1 + c^2 s_2^2)$, where

$$g_4(s_1) := 16 + 32s_1 + 64s_1^2 - 22s_1^3 - 23s_1^4 + 2s_1^5 + s_1^6.$$

We shall prove first that the points (s_1, s_2) in $\mathcal{H}_1$ for which $D(s_2) < 0$ are such that $0 \leq s_1 < 1.6$. Then we show that g_4 is positive in this interval. Since $V(s_1, s_2)$ increases with $s_1 \in [1, \bar{s}_1]$, decreases with $s_2 \in [0,1]$ and $V(1.6, \hat{s}_2) > V(1.6, 0.82) = 0.012116305 > 0$, we see that $\max\{s_1 : V(s_1, s_2) \leq 0, \forall 0 \leq s_2 \leq \hat{s}_2\} = \max\{s_1 : V(s_1, \hat{s}_2) \leq 0\} < 1.6$. Hence, it is sufficient to prove that $g_4(s_1) > 0$ for every $s_1 \in [1, 1.6]$. We first see that $g_4''(v+1) = -210 - 444v + 24v^2 + 160v^3 + 30v^4 < 0$ for every $v \in [0, 0.6]$, which means that $g_4(s_1)$ is a concave function on $[1, 1.6]$. Using that $g_4(1) = 70$ and $g_4(1.6) = 436624/15625 > 0$, we obtain $g_4(s_1) > 0$ for every $s_1 \in [1, 1.6]$ as desired. Hence, (ii) follows.

To prove (iii), we first show that $j_{2,0}^2 - j_2^2 > 0$ on $\mathcal{H}_1$. Denote by $(l_2, c_1) := (s_2^2, c^2)$. Recall that $(s_2, c) \in [0,1] \times [0, \sqrt{17/14}]$. Hence $(l_2, c_1) \in [0,1] \times [0, 17/14]$. We compute

$$\begin{aligned} j_{2,0}^2 - j_2^2 &= \frac{905}{4} + 364c_1 - 1728c_1^2 + 1152c_1^3 + 256c_1^4 + (701 - 5180c_1 + 7960c_1^2 \\ &\quad + 704c_1^3 + 256c_1^4)l_2 + (1 + 248c_1 + 832c_1^3 + 64c_1^4)l_2^2 - 1536c_1^4 l_2^3 - 256c_1^5 l_2^4 \\ &= \frac{l_2(1-l_2)}{2}\left(l_2(1-l_2)j_{21} + 4(1-l_2)^2 j_{22} + 4l_2^2 j_{23}\right) + (1-l_2)^4 j_{24} + l_2^4 j_{25}, \end{aligned}$$

where

$$\begin{aligned} j_{21} &= 6923 - 26216c_1 + 27024c_1^2 + 19712c_1^3 + 4736c_1^4, \\ j_{22} &= 803 - 1862c_1 + 524c_1^2 + 2656c_1^3 + 640c_1^4, \\ j_{23} &= 1505 - 6794c_1 + 8484c_1^2 + 4192c1^3 + 192c_1^4, \\ j_{24} &= 905/4 + 364c_1 - 1728c_1^2 + 1152c_1^3 + 256c_1^4, \\ j_{25} &= 3713/4 - 4568c_1 + 6232c_1^2 + 2688c_1^3 - 960c_1^4 - 256c_1^5. \end{aligned}$$

Let $v = 14c_1/17 \in [0,1]$. After manipulating these functions, we obtain

$$\begin{aligned} j_{21}(v) &= \frac{1-v}{49}\left(2v(96528 - 580911v + 1346231v^2) + 12000(3-10v)^2 + 231227(1-v)^3\right) \\ &\quad + \frac{145322731}{2401}v^4, \\ j_{22}(v) &= \frac{v(1-v)}{343}(326193 - 1061368v + 1671464v^2) + 803(1-v)^4 + \frac{13113085}{2401}v^4, \\ j_{23}(v) &= \frac{v(1-v)}{343}(58359 - 325948v + 3078524v^2) + 100(3-10v)^2 + 605(1-v)^4 + \frac{21099315}{2401}v^4, \\ j_{24}(v) &= \frac{v^2}{9604}(4183326 - 8388128v + 7465427v^2) + 100v(3-5v)^2 + 447v(1-v)^3 + \frac{905}{4}(1-v)^4, \\ j_{25}(v) &= \frac{v^2(1-v)^2}{98}(125841 + 401512v) + \frac{16643}{28}v(1-v)^4 + \frac{67796525}{9604}v^4(1-v) + 100(3-10v)^2 \\ &\quad + \frac{113}{4}(1-v)^5 + \frac{115642367}{67228}v^5, \quad \forall v \in [0,1]. \end{aligned}$$

One can check that $j_{21}, j_{22}, j_{23}, j_{24}$ and j_{25} are positive functions on $[0,1]$. Therefore, $j_{2,0}^2 - j_2^2 > 0$ for every $(s_2, c) \in [0,1] \times [0, \sqrt{17/14}]$. This proves (iii).

Finally, we prove (iv). From (9.13) and the definition of $D_\pm$, it is sufficient to prove that

$$d_+(s_2) := (1-s_2^2)(4+2s_2^2+s_2^4) \leq (1-s_2^2)^{1/2}(17-s_2^2)^{1/2} \leq d_-(s_2) := \frac{33}{8} - (2s_2^2+s_2^4),$$

for every $s_2 \in [0,1]$. Taking the square of the non-negative expressions above, we see that

$$\begin{aligned} &(1-s_2^2)(17-s_2^2) - d_+(s_2)^2 = (1-s_2^4)(1-2s_2^2+6s_2^4+2s_2^6+s_2^8) > 0, \\ &d_-(s_2)^2 - (1-s_2^2)(17-s_2^2) = \frac{s_2^4}{4}(71-287s_2^2+293s_2^4) + \frac{3s_2^2}{4}(11-20s_2^2)^2 \\ &\qquad\qquad + \frac{37s_2^2}{4}(1-s_2^2)^3 + (1-2s_2^2)^2 > 0, \quad \forall s_2 \in [0,1]. \end{aligned}$$

Item (iv) follows and the proof of Lemma 9.7 is complete. □

Assume that $D(s_2) < 0$. By Lemma 9.7, we obtain from (9.15) the following estimates

$$\begin{aligned} 16E_1 &= 16j_1c_{-1} - 16c^3D\cdot W_3 \\ &\geq \frac{59}{4}\cdot 16c_{-1} - 16c^3D\cdot W_3 \\ &= \frac{59}{4}\big(j_0 - 4cs_2^2(1+c^2s_2^2)j_2\big) - 16c^3D\cdot j_3j_2 - 16c^4D\cdot j_4 \\ &\geq \frac{59}{4}(j_0 - 4cs_2^2(1+c^2s_2^2)j_{2,0}) - c^3D_-\cdot j_3j_{2,0} - c^4D_+\cdot j_4 \\ &=: E_2(s_2,c), \quad (s_2,c) \in \Omega, \end{aligned} \tag{9.16}$$

where $j_{2,0}$ and $D_\pm$ are as in Lemma 9.7-(iii) and (iv). The first inequality follows from Lemmas 9.5 and 9.7-(ii). The last inequality follows from Lemma 9.7-(iv) and the relations $16c^3Dj_3j_2 \leq 16c^3Dj_3j_{2,0} \leq 16c^3D_-j_3j_{2,0}$ on Ω due to Lemma 9.7-(i), (ii) and (iii). Finally, by Lemma B.1, we have $16E_1(s_2,c) \geq E_2(s_2,c) > 0, \forall (s_2,c) \in \Omega$. Notice that in coordinates (s_2,c), Ω corresponds to the points in $\mathcal{H}_1$ lying in $[0,\hat{s}_2] \times [0, \sqrt{17/14}]$, or equivalently, $D \leq 0$. We conclude that $I_0 > 0$ on Ω. As mentioned before, it follows from (9.14) that $I_0(s_2,c) = O(s_2^4)$ uniformly in c near $s_2 = 0$. The proof of Lemma 9.6 is completed. □

We still need to show that the positivity of sectional curvatures of $\mathcal{M}^e_{\mu,E}$ and $\mathcal{M}^e_{\mu,E}$ implies that both subsets are strictly convex. This local-to-global property is proved in the next section.

9.3. **Sectional curvatures and strict convexity.** In this section, we discuss the local-to-global relation between positive sectional curvatures and strict convexity of a topologically embedded three-sphere in $\mathbb{R}^4$, possibly with a finite set $S \subset M$ of singularities. In particular, $\mathbb{R}^4 \setminus M$ bounds a compact set $B_M \subset \mathbb{R}^4$ and an unbounded open set $C_M = \mathbb{R}^4 \setminus B_M$. Assume that all sectional curvatures of $M \setminus S$ are positive. In particular, the principal curvatures of $M \setminus S$ have the same sign, and we may assume they are all positive. If M admits no singularity, i.e., M is embedded, then [68, Theorem 13.5] implies that M is strictly convex. This means that B_M is convex and every hyperplane tangent to M at a regular point has contact of order 1, i.e., if a curve $\alpha(t) \in M \setminus S$ satisfies $\alpha'(0) \neq 0$, then $\alpha''(0)$ has a non-trivial projection to the normal direction to M at $\alpha(0)$. The case $\#S = 1$ was considered in [64]. Let us consider the case $\#S = 2$.

Proposition 9.8. *Assume that $\#S = 2$, i.e., $S = \{S_+, S_-\}$. If all sectional curvatures of $M \setminus S$ are positive, then M is strictly convex.*

Proof. The convex hull of S is a line segment. Hence, we find a sufficiently large sphere S_R which contains M in its interior and is tangent to M at a regular point $p_0 \in M \setminus S$. We can choose a continuous normal vector N along $M \setminus S$ pointing towards C_M. In a rectangular system of coordinates near p_0, M is locally the graph of a strictly convex function f_{p_0} defined near the origin $0 \in T_{p_0}M$, with positive values in the direction of $-N$, and so that B_M corresponds to points above the graph of f_{p_0}. This is true since the same holds for S_R. This property is locally preserved and thus it must hold for every point in $M \setminus S$ since all sectional curvatures of $M \setminus S$ are positive and $M \setminus S$ is connected.

To prove that B_M is convex, it is enough to show that given $x, y \in M$, the line segment xy connecting x to y is contained in B_M. Fix $x \in M \setminus S$. Then $xy \subset B_M$ for every y sufficiently close to x by the local property of M near x proved before. Now notice that the two lines connecting x to S_+ and S_- intersect M in at most a countable subset $G \subset M$. This follows from the fact that $M \setminus S$ has positive curvature. It follows that $M \setminus G$ is connected, and we can take a continuous curve $\gamma(t) \in M \setminus G, t \in [0,1]$, connecting x to y. Then $x\gamma(t)$ does not intersect S. We claim that for each t, the line segment $x\gamma(t)$ is contained in B_M. In fact, we show that the set $I \subset [0,1]$ of t for which this property holds is open and closed in $[0,1]$. Notice that I is clearly closed in $[0,1]$ since B_M is compact. It must also be open since $x\gamma(t), t \in I$, must be transverse to M at both x and $\gamma(t)$ since otherwise, $x\gamma(t)$ contains points in C_M, a contradiction. Hence $xy \subset B_M$. The other cases for x and y follow from continuation. Hence, B_M is convex. Since the curvatures are positive, the intersection of each hyperplane tangent to $M \setminus S$ with S is a single point, and the contact has order 1. □

Theorems 9.1 and 9.4 imply that all sectional curvatures of $\mathcal{M}^e_{1/2,E}$ and $\mathcal{M}^m_{1/2,E}$ are positive for every $E \leq -2$. This local condition combined with Proposition 9.8 implies that both $\mathcal{M}^e_{1/2,E}$ and $\mathcal{M}^m_{1/2,E}$ are strictly convex for every $E \leq -2$. This completes the proof of Theorem 1.12.

10. Proof of Theorem 1.16

First we claim that for (μ, E) sufficiently close to $(-1/2, -2) = (-1/2, L_1(1/2))$, with $E < L_1(\mu)$, the $\mathbb{R}P^3$-components $\mathcal{M}^e_{\mu,E}$ and $\mathcal{M}^m_{\mu,E}$ are dynamically convex. Indeed, we know from Theorem 1.9 that there exists a neighborhood $\mathcal{U}_3 \subset \mathbb{R}^2 \times (\mathbb{R} \times \mathbb{R}/2\pi\mathbb{Z})$ of the singularities $S_\pm(1/2)$ corresponding to $l_1(1/2)$ so that, for (μ, E) sufficiently close to $(1/2, -2)$ and any contractible periodic orbit $P' \subset \mathcal{M}^e_{\mu,E} \cup \mathcal{M}^m_{\mu,E}$, that is not a cover of the Lyapunov orbit near $l_1(\mu)$ and intersects $\mathcal{U}_3$, the Conley-Zehnder index of P' is > 3. Because $\mathcal{M}^e_{1/2,-2}$ and $\mathcal{M}^m_{1/2,-2}$ are strictly convex, we also know that for (μ, E) sufficiently close to $(1/2, -2)$, with $E < L_1(\mu)$, any periodic orbit in $(\mathcal{M}^e_{\mu,E} \cup \mathcal{M}^m_{\mu,E}) \setminus \mathcal{U}_3$ has index ≥ 3.

Now since $\mathcal{M}^e_{1/2,E}$ and $\mathcal{M}^m_{1/2,E}$ are strictly convex for every $E < -2 = L_1(1/2)$, we find an open neighborhood $\mathcal{V} \subset \mathbb{R}^2$ of $\{1/2\} \times (-\infty, -2)$ so that both $\mathcal{M}^e_{\mu,E}$ and $\mathcal{M}^m_{\mu,E}$ are strictly convex for $(\mu, E) \in \mathcal{V}$. For $E \ll 0$, the neighborhood can be taken uniform since both $\mathcal{M}^e_{\mu,E}$ and $\mathcal{M}^m_{\mu,E}$ are uniformly strictly convex. We conclude that there exists $\epsilon_0 > 0$ so that for every $|\mu - 1/2| < \epsilon_0$ and $E < L_1(\mu)$, both $\mathcal{M}^e_{\mu,E}$ and $\mathcal{M}^m_{\mu,E}$ are dynamically convex. This proves (i). Items (ii) and (iii) directly follow from the main results in [39, 40, 42, 43].

Appendix A. Non-negative paths in $\mathrm{Sp}(2n)$

In this section, we prove the following statement, which is used to obtain a lower bound on the index of periodic orbits of Hamiltonians with a magnetic term in the kinetic energy.

Proposition A.1. *Given $M \in \mathrm{Sp}(2n)$, there exists a smooth path $\beta : [0,1] \to \mathrm{Sp}(2n)$ with $\beta(0) = I_{2n}, \beta(1) = M$ and $-J\dot\beta(t)\beta(t)^{-1} \geq 0,\ \forall t \in [0,1]$. Moreover, $n \leq \mu_{RS}(\beta) \leq 2n$.*

Proof. Define $M_1 \diamond M_2$ as the $(n_1+n_2) \times (n_1+n_2)$ matrix

$$M_1 \diamond M_2 = \begin{pmatrix} A_1 & 0 & B_1 & 0 \\ 0 & A_2 & 0 & B_2 \\ C_1 & 0 & D_1 & 0 \\ 0 & C_2 & 0 & D_2 \end{pmatrix}, \quad \text{where} \quad M_i = \begin{pmatrix} A_i & B_i \\ C_i & D_i \end{pmatrix}_{n_i \times n_i}, \ i = 1,2.$$

Since M is generically semi-simple (diagonalizable), Theorem 1.7.3 from [54] implies the existence of $P \in \mathrm{Sp}(2n)$ such that $M = P(N_1 \diamond \cdots \diamond N_k)P^{-1}$, where $N_i \in \mathrm{Sp}(2n_i), \sum_{i=1}^k n_i = n$, is one of the following matrices in normal form

$$R(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}, \quad D(\lambda) = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}, \quad D(\rho,\theta) = \begin{pmatrix} \rho R(\theta) & 0 \\ 0 & \rho^{-1} R(\theta) \end{pmatrix},$$

with $\theta \in \mathbb{R}/2\pi\mathbb{Z}$, $\lambda \in \mathbb{R}\backslash\{0\}$ and $\rho > 0$. If $N_i = R(\theta),\ 0 < \theta \leq 2\pi$, we define $N_{i,t} = R(t\theta), t \in [0,1]$. If $N_i = D(\lambda)$, we first define

$$N_{i,t} = \begin{cases} D(\lambda)R(\pi + \pi t) & \text{if } \lambda > 0, \\ D(-\lambda)R(t\pi) & \text{if } \lambda < 0, \end{cases} \quad \forall t \in [1/2, 1].$$

In particular, the eigenvalues of $N_{i,1/2}$ are $\{i, -i\}$. Then there exists $Q_\pm \in \mathrm{Sp}(2)$ such that $N_{i,1/2} = Q_\pm(\mp J)Q_\pm^{-1}$ with $\lambda = \pm|\lambda|$. We further define

$$N_{i,t} = \begin{cases} Q_+R(3\pi t)Q_+^{-1} & \text{if } \lambda > 0, \\ Q_-R(\pi t)Q_-^{-1} & \text{if } \lambda < 0, \end{cases} \quad \forall t \in [0, 1/2].$$

If $N_i = D(\rho,\theta), \theta \in [0, 2\pi)$, we first define

$$N_{i,t} = \begin{pmatrix} \rho I_2 & 0 \\ 0 & \rho^{-1}I_2 \end{pmatrix} \begin{pmatrix} R((2t-1)\theta) & 0 \\ 0 & R((2t-1)\theta) \end{pmatrix}, \quad \forall t \in [1/2, 1],$$

where $N_{i,1/2} = D(\rho) \diamond D(\rho) =: D(\rho)^{\diamond 2}$. Then we define

$$N_{i,t} = \begin{cases} (D(\rho)R((2t+1)\pi))^{\diamond 2}, & \text{if } t \in [1/4, 1/2], \\ (QR(6\pi t)Q^{-1})^{\diamond 2}, & \text{if } t \in [0, 1/4], \end{cases}$$

where $Q \in \mathrm{Sp}(2)$ satisfies $D(\rho)R(3\pi/2) = Q(-J)Q^{-1}$.

Since $R(at)$ satisfies $-J\frac{d}{dt}R(at)R(at)^{-1} = -aJ \cdot JR(at)R(-at) = aI_2 > 0$ for any $a > 0$, we conclude that $N_{i,t}$ is a piecewise positive regular path. The Robbin-Salamon index satisfies $1 \leq \mu_{\mathrm{RS}}(N_{i,t}|_{t\in[0,1]}) \leq 2$ for $N_{i,1} = R(\theta), D(\lambda)$ and $2 \leq \mu_{\mathrm{RS}}(N_{i,t}|_{t\in[0,1]}) \leq 4$ for $N_{i,1} = D(\rho,\theta)$. After reparametrizing each path $N_{i,t}, t \in [0,1]$, at $t = 1/4, 1/2$ by slowing down the parametrization at these points, and perturbing the end at $t = 1$, see [54], we obtain a smooth non-negative regular path $\beta(t) := P(N_{1,t} \diamond \cdots \diamond N_{k,t})P^{-1}, t \in [0,1]$, from I_{2n} to M satisfying $-J\dot{\beta}(t)\beta(t)^{-1} \geq 0$ and whose Robbin-Salamon index satisfies $n \leq \mu_{\mathrm{RS}}(\beta) \leq 2n$. □

Appendix B. Supplementary proofs

This section is devoted to the proof of the following lemma.

Lemma B.1. *The function $E_2 = E_2(s_2, c)$, given in* (9.16)*, is positive on* $[0,1] \times [0, \sqrt{17/14}]$.

Proof. Let $l_2 := s_2^2 \in [0,1]$, and rewrite E_2 as

$$\begin{aligned} 64E_2(l_2,c) = {} & E_3l_2^3(1-l_2) + h_1l_2^7(1-l_2) + h_2l_2^7 + h_3l_2^6 + h_4l_2^2(1-l_2)^4 + h_5l_2(1-l_2)^5 \\ & + h_6(1-l_2)^6 + 10000c^4(3-5l_2)^2(31(4-5c)^2l_2^4/50 + (3-4c)^2(1-l_2)l_2^2 \\ & + (1-2c)^4(1-l_2)l_2 + c^4l_2^4) + 80c^9l_2^6(1-l_2) + 256c^{16}l_2^8(1-l_2^2), \end{aligned} \tag{B.1}$$

for every $(l_2, c) \in [0,1] \times [0, \sqrt{17/14}]$, where

$$\begin{aligned} E_3(l_2,c) &:= k_1(c)l_2^2 + k_2(c)l_2 + k_3(c), \\ h_1(l_2,c) &:= 16c^{11}(1 + 32c - 4c^2 + 256c^3 - 8c^4 + 32c^5 + 16c^3(8+c^2)l_2) > 0, \\ h_2(c) &:= 8c^{10}(2368 - 151c - 128c^2 + 160c^3 - 512c^4 + 48c^5 - 256c^6) > 0, \\ h_3(c) &:= 6608 - 22538c + 37760c^2 + 43056c^3 - 471552c^4 + 1083859c^5 - 667488c^6, \\ & \quad - 22371c^7 + 53248c^8 - 57015c^9 + 52096c^{10} - 1459c^{11} - 896c^{12} + 5440c^{13} \\ & \quad - 3584c^{14} + 624c^{15} + 1088c^{16} \\ h_4(c) &:= 99120 - 111746c + 188800c^2 - 709972c^3 - 272752c^4 + 2377365c^5 \\ & \quad - 2809728c^6 + 2257746c^7 - 1074240c^8 + 129136c^9 + 7616c^{11}, \\ h_5(c) &:= 39648 - 22302c + 37760c^2 - 407878c^3 + 259440c^4 + 725190c^5 \\ & \quad - 2140416c^6 + 3019152c^7 - 1440000c^8 + 27200c^9, \end{aligned}$$

$$h_6(c) := 6608 - 89964c^3 + 76160c^4 + 34272c^5 + 15232c^7.$$

and

$$\begin{aligned}
k_1(c) := {} & 72688 - 111982c + 188800c^2 - 349002c^3 + 1552416c^4 - 3429694c^5 + 2538200c^6 \\
& - 768657c^7 + 642128c^8 + 53009c^9 + 33152c^{10} + 4267c^{11} - 896c^{12} + 3784c^{13} \\
& + 8704c^{14} - 128c^{15}, \\
k_2(c) := {} & -165200 + 223492c - 377600c^2 + 991280c^3 - 1742336c^4 + 3920764c^5, \\
& - 5570360c^6 + 5195743c^7 - 2587792c^8 - 319833c^9 + 33152c^{10} - 17470c^{11} \\
& + 2176c^{12} - 3088c^{13} - 1088c^{15}, \\
k_3(c) := {} & 132160 - 223964c + 377600c^2 - 539456c^3 + 779872c^4 - 2021770c^5 + 3175040c^6, \\
& - 3205287c^7 + 1464320c^8 + 236090c^9 - 80512c^{10} + 11504c^{11} - 5440c^{13}.
\end{aligned}$$

The functions $h_1(l_2, c)$ and $h_2(c)$ are clearly positive for $0 \le l_2 \le 1$ and $0 \le c \le \sqrt{17/14}$. Let $\beta := 6/5 - c$.. Then $\beta > 0$ for every $c \in [0, \sqrt{17/14}] \subset [0, 6/5]$. We rewrite the expressions for h_3, h_4, h_5 and h_6 to obtain

$$\begin{aligned}
h_3 = {} & \left(\frac{17638696}{625} + \frac{5496536}{125}c + \frac{4054798}{25}c^2 + \left(\frac{487824}{5} + 634791\beta\right)c^3\right)(\frac{2}{5} - c)^2\beta \\
& \frac{1799104}{3125}c\beta + \frac{2(46460648 - 2460625c)}{78125} + (332348 + 145849c + 50716c^2)c^4(1-c)^4 \\
& + (1380 + 1138c)c^{10}(1-c)^2 + 163c^{11} + (3584\beta + \frac{6}{5})c^{13} + 624c^{15} + 1088c^{16}, \\
h_4 = {} & c(\frac{7}{10} - c)^2\left(\frac{32498403269}{125000} + \beta\left(\frac{55874294}{3125} + \frac{183617569}{250}c + \frac{114118354}{125}c^2 + \frac{1436406}{25}c^2\beta\right.\right. \\
& \left.\left. + \left(\frac{2917616}{5} + 129136\beta\right)c^4\right)\right) + \left(99120 - \frac{3120663099669}{12500000}c + \frac{99820440049}{625000}c^2\right) + 7616c^{11}, \\
h_5 = {} & c(\frac{13}{20} - c)^2\left(\frac{3703539883}{40000} + \beta\left(\frac{92744181}{1000} + \frac{52681943}{100}c + \frac{3077836}{5}c^2 + 503940c^3\right.\right. \\
& \left.\left. + 1404640c^4\right) + 27200c^6\right) + \frac{22592124459}{80000000} + \frac{489006553c}{3200000} + \frac{3744646701}{50000}(\frac{29}{40} - c)^2, \\
h_6 = {} & c^2(\frac{3}{5} - c)^2\left(\frac{16305856}{125} + \frac{1268064}{25}c + \frac{91392}{5}c^2 + 15232c^3\right) + \frac{6039012}{125}c(\frac{13}{20} - c)^2 \\
& + \left(6608 - \frac{255148257}{12500}c + \frac{49515186}{3125}c^2\right), \quad c \in [0, 6/5].
\end{aligned}$$

One can easily check that h_3, h_3, h_5 and h_6 are positive on $[0, 6/5]$. Therefore, it is sufficient to prove that $E_3(l_2, c)$ is positive on $[0, 1] \times [0, 6/5]$.

Now, we rewrite k_1 and k_3 and check that both are positive on $[0, 6/5]$

$$
\begin{aligned}
k_1(c) = &\left(\frac{581229138}{15625} + 500000c^3 + \frac{63625694}{25}c^4\right)(\frac{7}{10} - c)^2 + \frac{473182277c}{125}(\frac{3}{10} - c)^2(\frac{9}{10} - c)^2 \\
&+ \left(\frac{948554347}{6250} + \frac{136241809}{1250}\beta\right)(\frac{2}{5} - c)^2\beta + \left(642128\beta^4 + \frac{19273323c}{5}(\frac{3}{5} - c)^2\right)c^2\beta^2 \\
&+ c^9(53009 + 33152c^1 + 4267c^2 - 896c^3 + 3784c^4 + 8704c^5 - 128c^6) + \frac{32652259}{156250} \\
&+ \frac{15311007}{10000}c > 0,
\end{aligned}
$$

$$
\begin{aligned}
k_3(c) = c^2(\frac{17}{20} - c)^2 &\left(\frac{2355466141019}{4000000} + \frac{170920469037}{400000}c + \left(\frac{3936935827}{5000} + \frac{401995653}{1000}\beta\right)c^2\right. \\
&\left.+ \frac{42279081}{25}c^4 + \left(\frac{13026}{5} + 80512\beta\right)c^5\right) + \frac{24462722037599}{160000000}c(\frac{4}{5} - c)^2 + 11504c^{11} \\
&- 5440c^{13} + \left(132160 - \frac{80453722037599c}{250000000} + \frac{314833837847093c^2}{1600000000}\right) > 0.
\end{aligned}
$$

Computing E_3 and the line segment $s_2 = 1$ and $c \in [0, 6/5]$, we obtain

$$
\begin{aligned}
E_3(1, c) &= k_1(c) + k_2(c) + k_3(c) \\
&= \left(39648 - \frac{28903204136204c}{244140625} + \frac{5470057737716c^2}{48828125}\right) + \frac{1773112492}{78125}c^3\beta^2 \\
&\quad + c^3\beta^3\left(\frac{16976668754}{78125} + \frac{11890235417}{15625}c + \frac{1721158044}{3125}c^2\right) \\
&\quad + c^8\beta\left(\frac{36159674}{625} + \frac{2825154}{125}c + \frac{174859}{25}c^2 + \frac{22064}{5}c^3 + 4744c^4\right) \\
&\quad + c(\frac{3}{5} - c)^2\left(\frac{160957143606}{9765625} + \frac{104765224976c}{390625}\right) \\
&\quad + 8704c^{14} - 1216c^{15} > 0, \quad \forall c \in [0, 6/5].
\end{aligned}
$$

Then we compare $E_3(l_2, c) - k_3(c)(1 - l_2)^2 - E_3(1, c)l_2^2 = l_2(1 - l_2)k_5(c)$, where

$$
\begin{aligned}
k_5(c) := &\ 99120 - 224436c + 377600c^2 - 87632c^3 - 182592c^4 - 122776c^5 + 779720c^6 \\
&- 1214831c^7 + 340848c^8 + 152347c^9 - 127872c^{10} + 5538c^{11} + 2176c^{12} \\
&- 13968c^{13} - 1088c^{15}.
\end{aligned}
$$

We compute

$$
\begin{aligned}
-k_5'(c) = &\left(224436 - \frac{39166298761c}{40000} + \frac{8584984187c^2}{8000}\right) + c^2(\frac{1}{2} - c)^2\left(\frac{296223209}{400}\right. \\
&\left.+ \frac{29197057}{20}c + \frac{2934884627}{2500}c^2 + \beta\left(\frac{433202652}{125}c^2 + \frac{5077647}{25}c^3\right) + \frac{29200428}{25}c^5\right) \\
&+ \frac{110596281c}{100}(\frac{9}{20} - c)^2 + c^9\beta\left(\frac{461262}{5} + 26112c\right) + 181584c^{12} + 16320c^{14} > 0,
\end{aligned}
$$

for every $c \in [0, 6/5]$. Therefore, $k_5(c)$ is a decreasing function on that interval. Since $k_5(4/5) = 14698.5 > 0$, we conclude that $k_5(c) > 0$ for every $c \in [0, 4/5]$. This implies that $E_3(l_2, c) > 0$ for every $(l_2, c) \in [0, 1] \times [0, 4/5]$. This implies that $E_3(l_2, c)$ is positive for every $(l_2, c) \in [0, 1] \times [0, 4/5]$. It remains to show that E_3 is positive on $[0, 1] \times [4/5, 6/5]$.

Since $k_1(c) > 0$ for every $c \in [0, 6/5]$, $E_3(l_2, c)$ is a convex function in l_2. For every fixed $c \in [4/5, 6/5]$, the minimum value of $E_3(\cdot, c)$ is given by

$$
E_3\left(-\frac{k_2}{2k_1}, c\right) = \frac{1}{4k_1(c)}(4k_1(c)k_3(c) - k_2(c)^2) =: \frac{1}{4k_1(c)}D_0(c).
$$

We shall prove that $D_0(c) > 0$ for every $c \in [4/5, 6/5]$. We consider $c = 2v/5 + 4/5$, where $v \in [0,1]$. Notice that $D_0(2v/5 + 4/5)$ is a polynomial in v of degree 30 whose coefficients are all rational numbers. Replacing all such coefficients a by $\lfloor a \rfloor$, we obtain a new polynomial $D_1(v)$ satisfying

$$\begin{aligned}
&D_0(2v/5+4/5) \geq D_1(v)\\
:=\ &1171163506 - 7471707255v + 16174395360v^2 + 8922925732v^3 - 44809560461v^4\\
&- 33799370597v^5 + 45096619475v^6 + 78595290662v^7 + 37015742898v^8 - 14346554736v^9\\
&- 29900538237v^{10} - 19491958537v^{11} - 6963317252v^{12} - 1208863632v^{13} + 45920428v^{14}\\
&+ 44662945v^{15} - 18645624v^{16} - 19608465v^{17} - 8327827v^{18} - 2388872v^{19} - 519800v^{20}\\
&- 89861v^{21} - 12669v^{22} - 1482v^{23} - 145v^{24} - 12v^{25} - v^{26} - v^{27} - v^{28} - v^{29} - v^{30},
\end{aligned}$$

for every $v \in [0,1]$. We can rewrite D_1 as

$$\begin{aligned}
D_1(v) = &\frac{24063486356113}{320000} + \frac{5194158621777v}{80000}\\
&+ (\frac{1}{2} - v)^2\big(\frac{14476579533567}{16000} + \frac{2258156665941}{32}(\frac{1}{5} - v)^2\big)\\
&+ (\frac{2}{5} - v)^2\big(\frac{820472989093}{800}(1-v) + \frac{587799303261}{8}v(\frac{3}{5} - v)^2\big)\\
&+ v^4(\frac{1}{2} - v)^2\big(\frac{419040213759}{8} + (1-v)\big(\frac{147822902909}{2} + \frac{299960460267}{2}v\\
&+ 129475910052v^2 + 57564677658v^3\big)\big) + v^{10}(1-v)(27664139421\\
&+ 8172180884v + 1208863632v^2 + 4931817v^4 + 49594762v^5 + 30949138v^6\\
&+ 11340673v^7 + 3012846v^8 + 623974v^9 + 104174v^{10} + 14313v^{11}) + 40988611v^{14}\\
&+ (1-v)v^{22}(1644 + 162v + 17v^2 + 5v^3 + 4v^4 + 3v^5 + 2v^6 + v^7) > 0, \quad \forall v \in [0,1].
\end{aligned}$$

Therefore, $D(c) > 0$ for every $c \in [4/5, 6/5]$ and thus $E_3(l_2, c) > 0$ on $[0,1] \times [4/5, 6/5]$. Together with the estimate for every $c \in [0, 4/5]$, we conclude that $E_3(l_2, c) > 0$ for every $(l_2, c) \in [0,1] \times [0, 6/5]$. Finally, (9.16) and the previous estimates for $h_1, \ldots, h_6$ show that $E_2(l_2, c) > 0$ for every $(l_2, c) \in [0,1] \times [0, \sqrt{17/14}]$. This finishes the proof of the lemma. ☐

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